Boson representations of one-dimensional scattering

2000 J. Phys. A: Math. Gen. 33 191

(http://iopscience.iop.org/0305-4470/33/1/312)

J. Phys. A: Math. Gen. **33** (2000) 191–225. Printed in the UK PII: S0305-4470(00)04305-5

Boson representations of one-dimensional scattering

Toru Miyazawa
Department of Physics, Gakushuin University, Tokyo 171-8588, Japan
E-mail: toru.miyazawa@gakushuin.ac.jp

Received 14 May 1999, in final form 19 October 1999

Abstract. A new description of one-dimensional scattering processes in terms of boson operators is presented, and the Schrödinger equation in a general form is analysed in this description on the basis of the factorization scheme. As an example of application, a low-energy expansion formula of the Green function is derived within the framework of this formalism.

1. Introduction

Scattering theory of the Schrödinger equation in one dimension has always been an active area of research since the foundation of quantum mechanics. In recent years, there has been considerable interest in this area with relation to the inverse scattering method for nonlinear problems [1, 2]. A particularly important issue in scattering theory is the analysis of the Green function. Asymptotic expansions for the Green function and related functions in high- and low-energy regions have been extensively studied by many researchers using various methods. (See [3–5] and references therein.) The aim of this paper is to provide another viewpoint to this old problem.

Here we study the structure of the Green function on the basis of a new description of scattering processes. We consider the one-dimensional Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi(x;t) = -\frac{\partial^2}{\partial x^2}\psi(x;t) + V_S(x)\psi(x;t) \tag{1.1}$$

or its stationary-state form

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + V_S(x)\psi(x) = E\psi(x). \tag{1.2}$$

Here $V_S(x)$ is the potential and E denotes the energy. We set the origin of the energy scale at the ground state level, so that the ground state has zero energy.

In our analysis we make use of factorization of the Schrödinger operator. It is well known that the Schrödinger equation (1.2) can be factorized as

$$-\left[\frac{\mathrm{d}}{\mathrm{d}x} + f(x)\right]\left[\frac{\mathrm{d}}{\mathrm{d}x} - f(x)\right]\psi(x) = E\psi(x) \tag{1.3}$$

where

$$V_S(x) = [f(x)]^2 + \frac{\mathrm{d}f(x)}{\mathrm{d}x}. \tag{1.4}$$

Let $\psi_0(x)$ be a solution of equation (1.2) with $E = 0$, such that $\psi_0(x) \geqslant 0$ for any x. This ψ_0 is the wavefunction of the ground state if it is a bound state. The function $f(x)$ satisfying (1.4) is expressed in terms of ψ_0 as

$$f(x) = \frac{1}{\psi_0(x)} \frac{\mathrm{d}}{\mathrm{d}x} \psi_0(x). \tag{1.5}$$

This kind of factorization, which has a fairly long history, has recently been extensively utilized in the context of supersymmetric quantum mechanics [6, 7]. Aside from the supersymmetry, the factorization is useful for studying general properties of the Schrödinger equation. The factorized equation (1.3) is much more manageable than (1.2), and we can construct explicit formulae for the Green function in terms of the function $f(x)$ given by (1.5). This amounts to expressing the Green function in terms of the wavefunction of the ground state.

Another important aspect of the factorization is its close relation with the Fokker–Planck equation [8–11]. The one-dimensional Fokker–Planck equation describing the diffusion process in a potential $V(x)$ has the form [12, 13]

$$\frac{\partial}{\partial t} P(x;t) = \frac{\partial^2}{\partial x^2} P(x;t) - 2 \frac{\partial}{\partial x}[f(x)P(x;t)] \tag{1.6}$$

where

$$f(x) = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}x} V(x). \tag{1.7}$$

The Schrödinger equation (1.1) is equivalent to the Fokker–Planck equation (1.6), with the correspondence

$$\psi = \mathrm{e}^{V(x)/2} P \qquad t \leftrightarrow -\mathrm{i}t \tag{1.8}$$

the function $f(x)$ in equation (1.6) being the same as that in equation (1.3). Therefore, a study of the factorized Schrödinger equation is also important for applications in nonequilibrium problems.

Throughout this paper we let $V(x)$ denote the Fokker–Planck potential, which is related to $f(x)$ by (1.7). From (1.5) and (1.7) we have

$$V(x) = -2 \log \psi_0(x) + \text{constant}. \tag{1.9}$$

(We may drop the constant in (1.9), since it is always cancelled in the final result.) For the time being, we assume that both $V(+\infty)$ and $V(-\infty)$ are finite, and that $V(x)$ converges to these limits sufficiently fast. (We shall see the specific meaning of this 'sufficiently fast' in section 9.) However, as demonstrated later, we can also deal with the cases where $V(x)$ tends to infinity at $x \to \pm\infty$.

The Green function $G_S(x, x'; t)$ for the Schrödinger equation (1.1) is defined as the solution of

$$\mathrm{i}\frac{\partial}{\partial t} G_S(x, x'; t) + \frac{\partial^2}{\partial x^2} G_S(x, x'; t) - V_S(x) G_S(x, x'; t) = \delta(x - x')\delta(t). \tag{1.10}$$

We shall deal with retarded Green functions, and so G_S satisfies the condition $G_S(x, x'; t) = 0$ for $t < 0$. Its Fourier transform

$$G_S(x, x'; E) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}Et} G_S(x, x'; t) \mathrm{d}t \tag{1.11}$$

is the Green function for the stationary-state equation (1.2), satisfying

$$\left[\frac{\partial^2}{\partial x^2} - V_S(x) + E \right] G_S(x, x'; E) = \delta(x - x'). \tag{1.12}$$

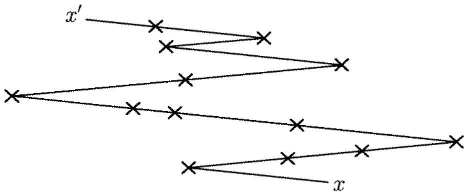

Figure 1. Schematic representation of the scattering process in one dimension. The rules for interpreting such diagrams are given in figure 2. The horizontal direction of the figure corresponds to the space coordinate. The vertical direction does not have any particular meaning. Such a diagram, when integrated over the positions of the crosses, amounts to the nth-order term in the Born expansion (1.19). (In this example, $n = 13$.)

Let $G_F(x, x'; t)$ denote the Green function of the Fokker–Planck equation (1.6), satisfying

$$\frac{\partial}{\partial t} G_F(x, x'; t) - \frac{\partial^2}{\partial x^2} G_F(x, x'; t) + 2\frac{\partial}{\partial x}[f(x) G_F(x, x'; t)] = \delta(x - x')\delta(t). \tag{1.13}$$

We define its Fourier transform

$$G_F(x, x'; \omega) \equiv \int_0^\infty e^{i\omega t} G_F(x, x'; t) dt. \tag{1.14}$$

Then we have the relation

$$G_F(x, x'; \omega) = -e^{-[V(x)-V(x')]/2} G_S(x, x'; E) \tag{1.15}$$

with $E = i\omega$. Thus, the Green function of the Fokker–Planck equation, which plays a fundamental role in diffusion problems, is obtained from G_S by analytic continuation to imaginary values of energy.

The free-particle Green function $G_0(x, x'; E)$ for the stationary Schrödinger equation is the solution of

$$\left(\frac{\partial^2}{\partial x^2} + E\right) G_0(x, x'; E) = \delta(x - x'). \tag{1.16}$$

Its explicit form is obtained as

$$G_0(x, x'; E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(x-x')}}{-p^2 + E + i\delta} dp = \frac{-i}{2k} e^{ik|x-x'|} \tag{1.17}$$

where

$$k = \sqrt{E}. \tag{1.18}$$

(Here δ is a positive infinitesimal quantity that ensures the 'retarded' boundary condition.) Using G_0, the Green function of equation (1.2) can be expanded in terms of V_S as

$$\begin{aligned} G_S(x, x'; E) &= G_0(x, x'; E) + \int_{-\infty}^{\infty} dy_1 G_0(x, y_1; E) V_S(y_1)\, G_0(y_1, x'; E) \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dy_1 dy_2 G_0(x, y_2; E) V_S(y_2) G_0(y_2, y_1; E) V_S(y_1) G_0(y_1, x'; E) + \cdots. \end{aligned} \tag{1.19}$$

This is the Born expansion. The Born expansion in one dimension is graphically represented by paths in one-dimensional space that connects the points x' and x, as in figures 1 and 2.

We may use (1.4) and write the Born expansion in terms of $f(x)$ rather than $V_S(x)$. The diagrammatic rules (figures 2(*b*)–(*d*)) become particularly simple when expressed in terms of $f(x)$. After an easy calculation (see appendix A), we obtain the following rules:

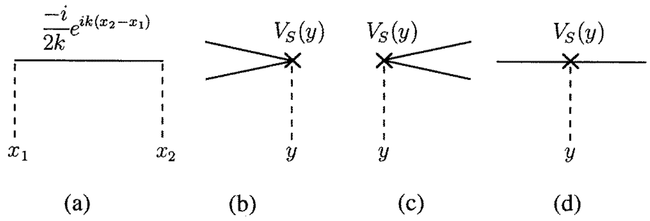

Figure 2. Rules for the diagrammatic representation.

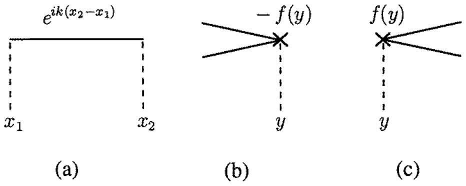

Figure 3. Diagrammatic rules expressed in terms of $f(x)$.

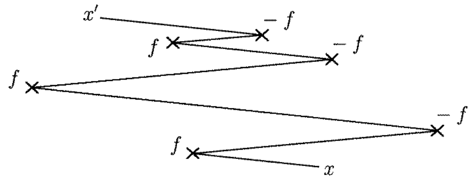

Figure 4. Graphical representation of the scattering process re-expressed in terms of f. The interaction takes place only at the turning points of the path.

(1) The left reflection at position y (figure 2(b)) gives a factor $-2\mathrm{i}kf(y)$.
(2) The right reflection at position y (figure 2(c)) gives a factor $+2\mathrm{i}kf(y)$.
(3) The forward scattering (figure 2(d)) vanishes.

The remarkable point is that there is no forward scattering, so that the interaction takes place only where the path changes its direction. These rules can be further simplified, since the factor $2\mathrm{i}k$ that comes from each scattering ((1) and (2) above) is cancelled by the factor $-\mathrm{i}/(2k)$ in G_0. Therefore, apart from an overall factor $-\mathrm{i}/(2k)$ in the final result, we can calculate the Green function with the diagrammatic rules given in figure 3. Now we treat $\mathrm{e}^{\mathrm{i}k(x_2-x_1)}$, instead of $-\mathrm{i}\mathrm{e}^{\mathrm{i}k(x_2-x_1)}/(2k)$, as the free propagator. The Green function is obtained by taking the sum of such diagrams as in figure 4, now with the rules of figure 3, and multiplying it with the overall factor $-\mathrm{i}/(2k)$. (Here the 'sum of diagrams' implies integrations over the positions of the turning points, as well as the summation over the number of turning points.)

The diagrammatic representation of the scattering process leads us in a natural way to a description in terms of boson operators [14]. We regard the space coordinate x as playing the role of time in usual quantum mechanics, and interpret the free propagator (figure 3(a)) as the propagation line of a boson. Since the free propagator connecting x' and x has the form $\exp[\mathrm{i}k(x-x')]$ for $x \geqslant x'$, we can see that the 'energy' of the free boson is $-k$. Therefore, the free propagation of the boson is described by the unperturbed 'Hamiltonian'

$$H_0 = -ka^\dagger a \tag{1.20}$$

where a and $a^\dagger$ are boson annihilation–creation operators satisfying the commutation relation

$$[a, a^\dagger] = 1. \tag{1.21}$$

We interpret the scattering vertices (figures 3(b) and (c)) as the pair annihilation and pair creation of bosons. The vertex in figure 3(b), which is interpreted as the pair annihilation, is represented by the operator $-\frac{1}{2}faa$. (The $\frac{1}{2}$ is a symmetry factor.) Similarly, figure 3(c) is represented by $\frac{1}{2}fa^\dagger a^\dagger$. So the scattering is described by the interaction Hamiltonian

$$H'(x) = -\frac{\mathrm{i}}{2}f(x)(aa - a^\dagger a^\dagger). \tag{1.22}$$

As in usual quantum mechanics, we define the evolution operator U as the solution of

$$\mathrm{i}\frac{\partial}{\partial x}U(x, x_0; k) = [H_0 + H'(x)]U(x, x_0; k) \tag{1.23}$$

with the initial condition $U(x = x_0; k) = 1$. Using this evolution operator, we define the complete propagator as

$$G(x, x'; k) \equiv \frac{\langle 0|U(\infty, x; k)(a + a^\dagger)U(x, x'; k)(a + a^\dagger)U(x', -\infty; k)|0\rangle}{\langle 0|U(\infty, -\infty; k)|0\rangle} \tag{1.24}$$

where $|0\rangle$ is the vacuum state satisfying $a|0\rangle = 0$. (Hereafter we assume $x \geqslant x'$ without loss of generality.) It is obvious that this G gives the sum of all such diagrams as the one in figure 4, with the rules of figure 3. The quantity $\langle 0|U(\infty, -\infty; k)|0\rangle$ in the denominator cancels the contribution from disconnected diagrams. The operator $a + a^\dagger$ is assigned to the points x and x', since an endpoint of a path corresponds to either creation or annihilation of a boson line. Therefore, we find that the Green function of the Schrödinger equation is obtained as

$$G_S(x, x'; E) = \frac{-\mathrm{i}}{2k}G(x, x'; k) \qquad k = \sqrt{E}. \tag{1.25}$$

Let us note that we may shift the unperturbed Hamiltonian (1.20) by an arbitrary constant. Adding a constant c to H_0 gives rise to an extra factor $\exp[-\mathrm{i}c(x_2 - x_1)]$ in front of the evolution operator $U(x_2, x_1; k)$. Such factors are cancelled between the denominator and the numerator of (1.24), and so expression (1.24) remains unchanged. For convenience in our future discussion, we replace (1.20) by

$$H_0 = -ka^\dagger a - \frac{k}{2} = -\frac{k}{2}(aa^\dagger + a^\dagger a). \tag{1.26}$$

The evolution equation (1.23) now takes the form

$$\frac{\partial}{\partial x}U(x, x_0; k) = \tfrac{1}{2}[\mathrm{i}k(aa^\dagger + a^\dagger a) - f(x)(aa - a^\dagger a^\dagger)]U(x, x_0; k). \tag{1.27}$$

The Born expansion is essentially a high-energy expansion. Although (1.24) is originally derived from the Born expansion, we can turn it into a low-energy expansion by a symmetry transformation. This is a remarkable feature of expression (1.24), and we shall study it thoroughly in this paper.

The boson representation depicted above is the 'old' boson picture already discussed in previous papers [14, 15]. Although it is useful for understanding many properties of scattering processes, it is not necessarily the most suitable representation for actual calculations. In this paper we introduce a different representation in terms of boson operators of a different kind, which will provide a new viewpoint on scattering processes in general.

We begin our analysis with the old boson representation. We study the structure of the propagator in sections 3 and 4, starting from the expression (1.24). In section 5 we introduce a new representation in terms of another kind of boson operator. In section 6, an expression of the propagator is derived in this representation. As an application of this formalism, in sections 7 and 8 we derive a low-energy expansion formula of the Green function [16] by using this new boson representation.

2. Scattering coefficients

Before getting into the main thread, let us briefly remark upon the scattering coefficients (transmission and reflection coefficients). Although we do not develop a detailed discussion on the scattering coefficients in this paper, they are very important quantities in describing scattering processes. So it is worthwhile to notice here how they can be dealt with in our formalism.

We define the scattering coefficients for a finite interval (x_1, x_2) as follows [15]. We consider the Fokker–Planck potential $\bar{V}(x)$, which is the same as $V(x)$ within the interval (x_1, x_2) and constant outside:

$$\bar{V}(x) \equiv \begin{cases} V(x_1) & (x < x_1) \\ V(x) & (x_1 \leqslant x \leqslant x_2) \\ V(x_2) & (x_2 < x). \end{cases} \tag{2.1}$$

The corresponding Schrödinger potential is

$$\bar{V}_S(x) \equiv \bar{f}^2 + \frac{\mathrm{d}\bar{f}}{\mathrm{d}x} \qquad \bar{f}(x) \equiv -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\bar{V}(x). \tag{2.2}$$

(The function $\bar{V}(x)$ defined by (2.1) is not, in general, smooth at $x = x_1$ and $x = x_2$. Therefore $\bar{f}(x)$ is discontinuous, and $\bar{V}_S(x)$ includes delta functions at these points†.) Since $\bar{V}_S(x) = 0$ outside the interval (x_1, x_2), the Schrödinger equation

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \bar{V}_S\psi = k^2\psi \tag{2.3}$$

has two independent solutions of the form

$$\psi_1(x) = \begin{cases} \mathrm{e}^{\mathrm{i}k(x-x_1)} + R_l(x_2, x_1)\mathrm{e}^{-\mathrm{i}k(x-x_1)} & (x < x_1) \\ \tau(x_2, x_1)\mathrm{e}^{\mathrm{i}k(x-x_2)} & (x_2 < x) \end{cases} \tag{2.4a}$$

$$\psi_2(x) = \begin{cases} \tau(x_2, x_1)\mathrm{e}^{-\mathrm{i}k(x-x_1)} & (x < x_1) \\ \mathrm{e}^{-\mathrm{i}k(x-x_2)} + R_r(x_2, x_1)\mathrm{e}^{\mathrm{i}k(x-x_2)} & (x_2 < x). \end{cases} \tag{2.4b}$$

This defines the right reflection coefficient R_r, left reflection coefficient R_l, and the transmission coefficient τ for the interval (x_1, x_2). The connection to the standard scattering coefficients, which are defined with regard to the asymptotic behaviour at infinity, is explained in [15].

These coefficients can be expressed in the boson representation as‡

$$\tau(x_2, x_1) = \frac{\langle 0|aU(x_2, x_1)a|0\rangle}{\langle 0|U(x_2, x_1)|0\rangle} \tag{2.5a}$$

$$R_r(x_2, x_1) = \frac{\langle 0|aaU(x_2, x_1)|0\rangle}{\langle 0|U(x_2, x_1)|0\rangle} \tag{2.5b}$$

$$R_l(x_2, x_1) = \frac{\langle 0|U(x_2, x_1)a^\dagger a^\dagger|0\rangle}{\langle 0|U(x_2, x_1)|0\rangle}. \tag{2.5c}$$

Namely, the transmission coefficient $\tau(x_2, x_1)$ is the sum over the paths that penetrate the interval (x_1, x_2) without leaving it. The right reflection coefficient $R_r(x_2, x_1)$ is the sum over the paths, also restricted within (x_1, x_2), that start from and return to x_2, the right edge of the interval. Similarly, R_l is obtained as the sum over the paths that start from and return to the left edge.

We can treat these scattering coefficients in the same way as the Green function. Let us generalize the definition of the propagator (1.24) as

$$G(x, x'; z, z') \equiv \frac{\langle 0|U(z, x)(a + a^\dagger)U(x, x')(a + a^\dagger)U(x', z')|0\rangle}{\langle 0|U(z, z')|0\rangle} \tag{2.6}$$

† In fact, the scattering coefficients are well defined even if $\bar{V}(x)$ has discontinuities, although $\bar{V}_S(x)$ is then rather problematic. In such cases, we may regard $\bar{V}(x)$ as a limit of continuous functions.

‡ See [14, 15] and references therein.

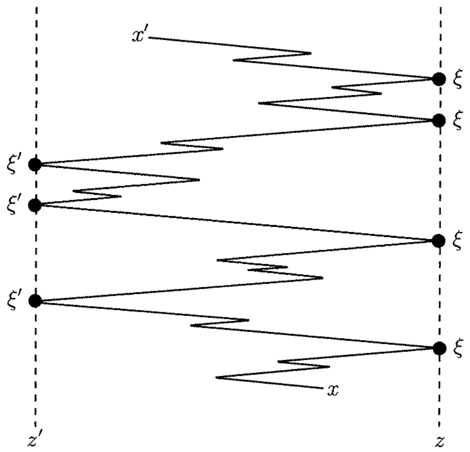

Figure 5. Diagrammatic interpretation of the generalized propagator $\bar{G}(x, x'; z, z'; \xi, \xi')$. (We omit the cross representing the interaction, which takes place wherever the path changes its direction.)

where z and z' are real numbers such that $z' \leqslant x' \leqslant x \leqslant z$. (Here and hereafter we omit to write the argument k.) This is the sum over the paths connecting x' and x, now restricted within the interval (z', z). The original propagator is recovered by setting $z = +\infty$ and $z' = -\infty$. From (2.6) and (2.5) we can see that

$$G(x_2, x_1; x_2, x_1) = \tau(x_2, x_1) \tag{2.7a}$$

$$G(x_2, x_2; x_2, x_1) = 1 + R_r(x_2, x_1) \tag{2.7b}$$

$$G(x_1, x_1; x_2, x_1) = 1 + R_l(x_2, x_1). \tag{2.7c}$$

Thus the propagator describes the scattering coefficients as well as the Green function.

3. General formalism

It is convenient to further generalize (2.6) and define

$$\bar{G}(x, x'; z, z'; \xi, \xi') \equiv \frac{\langle 0|\mathrm{e}^{\xi aa/2} U(z, x)(a + a^\dagger) U(x, x')(a + a^\dagger) U(x', z') \mathrm{e}^{\xi' a^\dagger a^\dagger/2}|0\rangle}{\langle 0|\mathrm{e}^{\xi aa/2} U(z, z') \mathrm{e}^{\xi' a^\dagger a^\dagger/2}|0\rangle} \tag{3.1}$$

with new parameters ξ and ξ'. This $\bar{G}$ is the sum of all such diagrams as in figure 5. Corresponding to the newly inserted operators $\mathrm{e}^{\xi aa/2}$ and $\mathrm{e}^{\xi' a^\dagger a^\dagger/2}$, these diagrams include additional scattering vertices at the points z and z'. Each scattering (reflection) at z and z' gives a factor ξ and ξ', respectively. Of course, (3.1) reduces to (2.6) by setting $\xi = \xi' = 0$. From (1.25) and (2.7) we find that the Green function and the scattering coefficients are obtained from $\bar{G}$ as

$$G_S(x, x') = \frac{-\mathrm{i}}{2k} \bar{G}(x, x'; \infty, -\infty; 0, 0) \tag{3.2a}$$

$$\tau(x_2, x_1) = \bar{G}(x_2, x_1; x_2, x_1; 0; 0) \tag{3.2b}$$

$$R_r(x_2, x_1) = \bar{G}(x_2, x_2; x_2, x_1; 0, 0) - 1 \tag{3.2c}$$

$$R_l(x_2, x_1) = \bar{G}(x_1, x_1; x_2, x_1; 0, 0) - 1. \tag{3.2d}$$

The generalized form (3.1) proves suitable for studying the symmetry structure of the propagator, as we shall see. Moreover, there is also a practical merit. So far we have been assuming that the Fokker–Planck potential $V(x)$ is finite at both $x \to +\infty$ and $-\infty$. In such cases, the original Green function G is recovered from $\bar{G}$ by setting $\xi = \xi' = 0$ and then taking the limit $z \to +\infty$, $z' \to -\infty$. However, including the parameters ξ and ξ' enables us to deal with other cases as well. Note that the value $\xi = 1$ corresponds to a totally reflecting wall at position z. So if $V(x)$ diverges to $+\infty$ at $x \to \infty$, we may set $\xi = 1$ (instead of $\xi = 0$) before letting $z \to \infty$. Indeed, this procedure proves to give the correct low-energy expression of the Green function, provided that $V(x)$ diverges sufficiently fast. (We shall comment on this condition in section 9.) On the other hand, $\xi = -1$ describes a totally absorbing wall. Therefore, if $V(x)$ goes to $-\infty$ at $x \to \infty$, it is appropriate to set $\xi = -1$ before $z \to \infty$. Similarly, we may set $\xi' = 0$, $+1$, or -1, corresponding to the cases $V(-\infty) =$ finite, $+\infty$, or $-\infty$, respectively, before taking the limit $z' \to -\infty$. Thus, using the generalized expression (3.1), we can deal with the cases where $V(x)$ is infinite at $x \to +\infty$ or $x \to -\infty$, or both.

Expression (3.1) can be written in a purely algebraic form as follows. (See [14] for details.) First, we define

$$J_+ = -\tfrac{1}{2}a^\dagger a^\dagger \qquad J_- = \tfrac{1}{2}aa \qquad J_3 = \tfrac{1}{4}(aa^\dagger + a^\dagger a) \tag{3.3a}$$

$$Q_+ = \frac{1}{\sqrt{2}}a^\dagger \qquad Q_- = \frac{1}{\sqrt{2}}a. \tag{3.3b}$$

Then J_3, $J_\pm$, and $Q_\pm$ satisfy

$$J_+ = -Q_+^2 \qquad J_- = Q_-^2 \qquad J_3 = \tfrac{1}{2}(Q_+Q_- + Q_-Q_+) \tag{3.4a}$$

$$[J_3, Q_+] = \tfrac{1}{2}Q_+ \qquad [J_3, Q_-] = -\tfrac{1}{2}Q_- \tag{3.4b}$$

where $[A, B] = AB - BA$. Operators satisfying relations (3.4) constitute a Lie superalgebra [17,18]. (This superalgebra bears no relation to the supersymmetry mentioned in section 1, which is associated with the factorization of the Schrödinger operator.) From (3.4) we can derive the remaining commutation relations among these operators:

$$[J_3, J_+] = J_+ \qquad [J_3, J_-] = -J_- \qquad [J_+, J_-] = 2J_3 \tag{3.5a}$$

$$[J_-, Q_+] = Q_- \qquad [J_+, Q_-] = Q_+ \qquad [J_+, Q_+] = [J_-, Q_-] = 0. \tag{3.5b}$$

Note that equations (3.5*a*) are nothing but the $SU(2)$ (or $SL(2, \boldsymbol{C})$) commutation relations. Just like the usual angular momentum operators, we use the notation

$$J_1 = (J_+ + J_-)/2 \qquad J_2 = -\mathrm{i}(J_+ - J_-)/2. \tag{3.6}$$

Next, we write

$$(\Psi, \Phi) = \langle\Psi|\Phi\rangle \tag{3.7}$$

where the right-hand side denotes the ordinary inner product in the Fock space. (Perhaps it would be more appropriate to write (3.7) as, for example, $(\Psi_m, \Psi_n) = \langle m|n\rangle$. But the meaning of (3.7) is clear.) Then, from (3.3) it is obvious that

$$(\Psi, Q_\pm\Phi) = (Q_\mp\Psi, \Phi) \tag{3.8}$$

with arbitrary states Ψ and Φ. Finally, we write Ψ_0 in place of $|0\rangle$. Namely,

$$\Psi_0 = |0\rangle. \tag{3.9}$$

This Ψ_0 is the lowest state, in the sense that it is annihilated by the operator Q_-:

$$Q_-\Psi_0 = 0. \tag{3.10}$$

Using these notations, we can rewrite (3.1) as

$$\bar{G}(x, x'; z, z'; \xi, \xi') = \frac{1}{2}\frac{(\Psi_0, \mathrm{e}^{\xi J_-}U(z, x)(Q_+ + Q_-)U(x, x')(Q_+ + Q_-)U(x', z')\mathrm{e}^{-\xi' J_+}\Psi_0)}{(\Psi_0, \mathrm{e}^{\xi J_-}U(z, z')\mathrm{e}^{-\xi' J_+}J_3\Psi_0)}. \tag{3.11}$$

(We have used $J_3|0\rangle = \frac{1}{4}|0\rangle$ and inserted a J_3 in the denominator on the right-hand side.) The evolution equation (1.27) can be rewritten as

$$\frac{\partial}{\partial x}U(x, x_0) = 2[\mathrm{i}kJ_3 - f(x)J_1]U(x, x_0). \tag{3.12}$$

In other words, the operator U that appears in (3.11) is obtained as the solution of (3.12) with the initial condition $U(x = x_0) = 1$ (identity operator).

It may seem that (3.11) is merely (3.1) rewritten in a different style. However, as it turns out, (3.11) is a more general expression. As shown in appendix B, expression (3.11) holds without reference to (3.3), (3.7) or (3.9). Namely, (3.11) holds in any representation as long as the algebraic relations (3.4), (3.8) and (3.10) are satisfied †. So we may forget (3.3), (3.7) and (3.9); all we need is (3.4), (3.8), (3.10) and (3.12). Expression (3.1) in terms of the boson operators is merely a specific representation of the general expression (3.11).

We may use (3.11) in any representation. If we are presented with a specific set of operators J_3, $J_\pm$, and $Q_\pm$ satisfying relations (3.4), we only need to define an inner product in the representation space such that condition (3.8) may be satisfied. Then from (3.11) and (3.12) we can obtain the expression of $\bar{G}$ in that particular representation.

As a preparation for the analysis in subsequent sections, let us describe here the structure of the evolution equation (3.12) in a general way. This equation can be expressed as

$$\frac{\partial}{\partial x}U(x, x_0) = 2\left(\frac{\mathrm{d}X}{\mathrm{d}x}J_3 + \frac{\mathrm{d}Y}{\mathrm{d}x}J_1\right)U(x, x_0) \tag{3.13}$$

with

$$X(x) \equiv \mathrm{i}kx \qquad Y(x) \equiv \tfrac{1}{2}V(x). \tag{3.14}$$

(Recall (1.7) for the definition of $V(x)$.) It is obvious that equation (3.13) has a symmetry structure. Let us define

$$\begin{pmatrix} X_\theta \\ Y_\theta \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} X \\ Y \end{pmatrix} \tag{3.15a}$$

$$\begin{pmatrix} J_{3,\theta} \\ J_{1,\theta} \end{pmatrix} \equiv \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} J_3 \\ J_1 \end{pmatrix}. \tag{3.15b}$$

Obviously the evolution equation (3.13) is covariant with respect to the transformation $(X, Y) \to (X_\theta, Y_\theta)$, $(J_3, J_1) \to (J_{3,\theta}, J_{1,\theta})$. Namely, we can rewrite equation (3.13) as

$$\frac{\partial}{\partial x}U(x, x_0) = 2\left(\frac{\mathrm{d}X_\theta}{\mathrm{d}x}J_{3,\theta} + \frac{\mathrm{d}Y_\theta}{\mathrm{d}x}J_{1,\theta}\right)U(x, x_0) \tag{3.16}$$

with arbitrary θ.

In addition to (3.15b), let us define $J_{2,\theta} \equiv J_2$ and also $J_{\pm,\theta} \equiv J_{1,\theta} \pm \mathrm{i}J_{2,\theta}$. The transformation $(J_1, J_2, J_3) \to (J_{1,\theta}, J_{2,\theta}, J_{3,\theta})$ is a rotation around the 2-axis, and so we can write

$$J_{a,\theta} = P(\theta)J_aP(-\theta) \qquad (a = 1, 2, 3, \text{ or } \pm) \tag{3.17}$$

with

$$P(\theta) \equiv \mathrm{e}^{-\mathrm{i}\theta J_2}. \tag{3.18}$$

It can be shown‡ that this rotation operator $P(\theta)$ is expressed in terms of J_3 and $J_\pm$ as

$$P(\theta) = \mathrm{e}^{\tan(\theta/2)J_-}[\cos(\theta/2)]^{2J_3}\mathrm{e}^{-\tan(\theta/2)J_+}. \tag{3.19}$$

† Actually, the condition $(\Psi_0, Q_+\cdots) = 0$ is sufficient instead of (3.8). However, it is convenient to impose the requirement (3.8) in order to have a symmetrical description.
‡ See footnote 14 of [15].

(Here $[\cos(\theta/2)]^{2J_3} = \mathrm{e}^{2\log[\cos(\theta/2)]J_3}$.) Expression (3.19) turns out to be more useful than (3.18) for actual calculations.

Equation (3.16) is a generalized form of (1.27). As we did with equation (1.27), we shall regard the term $2\frac{\mathrm{d}X_\theta}{\mathrm{d}x}J_{3,\theta}$ in (3.16) as describing the unperturbed evolution, and $2\frac{\mathrm{d}Y_\theta}{\mathrm{d}x}J_{1,\theta}$ as the 'interaction'. Then it is convenient to switch over to the the interaction picture, as in usual quantum mechanics. If the interaction term were absent in (3.16), the evolution operator would behave as $\mathrm{e}^{2X_\theta J_{3,\theta}}$. So we can define the evolution operator in the interaction picture as

$$U_I(x_2, x_1; \theta) \equiv \mathrm{e}^{-2X_\theta(x_2)J_{3,\theta}} U(x_2, x_1) \mathrm{e}^{2X_\theta(x_1)J_{3,\theta}}. \tag{3.20}$$

(Note that the interaction picture depends on the frame angle θ.) From (3.16) and (3.20) we have

$$\frac{\partial}{\partial x} U_I(x, x_0; \theta) = 2\frac{\mathrm{d}Y_\theta}{\mathrm{d}x} \mathrm{e}^{-2X_\theta(x)J_{3,\theta}} J_{1,\theta} \mathrm{e}^{2X_\theta(x)J_{3,\theta}} U_I(x, x_0; \theta). \tag{3.21}$$

The operators $J_{\pm,\theta}$ and $J_{3,\theta}$ satisfy the same commutation relations as $J_\pm$ and J_3 (equations (3.5a)). By using these commutation relations, we can show that

$$\mathrm{e}^{-AJ_{3,\theta}} J_{\pm,\theta} \mathrm{e}^{AJ_{3,\theta}} = \mathrm{e}^{\mp A} J_{\pm,\theta} \tag{3.22}$$

for an arbitrary complex number A. Therefore, (3.21) becomes

$$\frac{\partial}{\partial x} U_I(x, x_0; \theta) = \frac{\mathrm{d}Y_\theta}{\mathrm{d}x} (\mathrm{e}^{-2X_\theta(x)} J_{+,\theta} + \mathrm{e}^{2X_\theta(x)} J_{-,\theta}) U_I(x, x_0; \theta). \tag{3.23}$$

In correspondence with (3.17), let us also define

$$Q_{\pm,\theta} = P(\theta) Q_\pm P(-\theta). \tag{3.24}$$

From (3.4*b*) and (3.5*b*) it follows that

$$Q_\pm A^{J_3} = A^{J_3 \pm 1/2} Q_\pm \qquad [\mathrm{e}^{AJ_\mp} Q_\pm] = A Q_\mp \mathrm{e}^{AJ_\mp} \tag{3.25}$$

where A is an arbitrary complex number. Substituting (3.19) into (3.24), and using (3.25), we obtain

$$\begin{pmatrix} Q_{+,\theta} \\ Q_{-,\theta} \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} Q_+ \\ Q_- \end{pmatrix}. \tag{3.26}$$

The commutation relations (3.4*b*) hold with J_3 and $Q_\pm$ replaced by $J_{3,\theta}$ and $Q_{\pm,\theta}$. Hence we have, just like (3.22),

$$\mathrm{e}^{-AJ_{3,\theta}} Q_{\pm,\theta} \mathrm{e}^{AJ_{3,\theta}} = \mathrm{e}^{\mp A/2} Q_{\pm,\theta}. \tag{3.27}$$

In this section we derived some general expressions in a rather abstract manner. We shall make use of these expressions in our subsequent studies.

4. Low-energy expansion in the path representation

The rotation introduced in the previous section is practically important because it is the transformation that turns the Born expansion into a low-energy expansion, as we shall see in this section. To describe the low-energy expansion in terms of boson operators, let us extend the boson picture to the rotated frame.

In the boson representation, the rotation operator (3.19) takes the form

$$P(\theta) = \mathrm{e}^{\tan(\theta/2)aa/2} [\cos(\theta/2)]^{(a^\dagger a + aa^\dagger)/2} \mathrm{e}^{\tan(\theta/2)a^\dagger a^\dagger/2}. \tag{4.1}$$

We define the boson operators in the rotated frame as

$$a_\theta \equiv P(\theta) a P(-\theta) \qquad a_\theta^\dagger \equiv P(\theta) a^\dagger P(-\theta). \tag{4.2}$$

From (3.26) and (3.3*b*) we have

$$\begin{pmatrix} a_\theta \\ a_\theta^\dagger \end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{4.3}$$

Obviously a_θ and $a_\theta^\dagger$ satisfy the boson commutation relation $[a_\theta, a_\theta^\dagger] = 1$. The operators $J_{a,\theta}$ (equation (3.17)) can be written in terms of a_θ and $a_\theta^\dagger$ as

$$J_{+,\theta} = -\tfrac{1}{2} a_\theta^\dagger a_\theta^\dagger \qquad J_{-,\theta} = \tfrac{1}{2} a_\theta a_\theta \qquad J_{3,\theta} = \tfrac{1}{4}(a_\theta a_\theta^\dagger + a_\theta^\dagger a_\theta). \tag{4.4}$$

We define the ket and bra representing the vacuum state in the rotated frame as

$$|0;\theta\rangle \equiv P(\theta)|0\rangle \qquad \langle 0;\theta| \equiv \langle 0|P(-\theta). \tag{4.5}$$

They satisfy

$$a_\theta|0;\theta\rangle = 0 \qquad \langle 0;\theta|a_\theta^\dagger = 0. \tag{4.6}$$

In the original sense, the bra conjugate to the ket $P(\theta)|0\rangle$ is $\langle 0|P(\theta)$ (assuming that θ is real), and not $\langle 0|P(-\theta)$. By defining $|0;\theta\rangle$ and $\langle 0;\theta|$ as in (4.5), we are changing the correspondence between the ket and the bra. This means that we are also changing the definition of the inner product in passing to the rotated frame. The operators a_θ and $a_\theta^\dagger$ are still the adjoint of each other with respect to this re-defined inner product.

As shown in appendix C, we have the relations

$$\mathrm{e}^{\xi' a^\dagger a^\dagger/2}|0\rangle = \left(\cos\frac{\theta}{2} + \xi' \sin\frac{\theta}{2}\right)^{-1/2} \mathrm{e}^{\xi'_{-\theta} a_\theta^\dagger a_\theta^\dagger/2}|0;\theta\rangle \tag{4.7a}$$

$$\langle 0|\mathrm{e}^{\xi a a/2} = \left(\cos\frac{\theta}{2} - \xi \sin\frac{\theta}{2}\right)^{-1/2} \langle 0;\theta|\mathrm{e}^{\xi_\theta a_\theta a_\theta/2} \tag{4.7b}$$

for arbitrary θ, where

$$\xi_\theta \equiv \frac{\tan(\theta/2) + \xi}{1 - \xi\tan(\theta/2)} \qquad \xi'_{-\theta} \equiv \frac{-\tan(\theta/2) + \xi'}{1 + \xi'\tan(\theta/2)}. \tag{4.8}$$

Therefore, (3.1) may be written with arbitrary θ as

$$\bar{G} = \frac{\langle 0;\theta|\mathrm{e}^{\xi_\theta a_\theta a_\theta/2} U(z,x)(a+a^\dagger)U(x,x')(a+a^\dagger)U(x',z')\mathrm{e}^{\xi'_{-\theta} a_\theta^\dagger a_\theta^\dagger/2}|0;\theta\rangle}{\langle 0;\theta|\mathrm{e}^{\xi_\theta a_\theta a_\theta/2} U(z,z')\mathrm{e}^{\xi'_{-\theta} a_\theta^\dagger a_\theta^\dagger/2}|0;\theta\rangle} \tag{4.9}$$

where

$$a + a^\dagger = \left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right) a_\theta + \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) a_\theta^\dagger. \tag{4.10}$$

The evolution equation (3.16) is written in terms of a_θ and $a_\theta^\dagger$ as

$$\begin{aligned} \frac{\partial}{\partial x}U(x,x_0) &= \frac{1}{2}\left[\frac{\mathrm{d}X_\theta}{\mathrm{d}x}(a_\theta^\dagger a_\theta + a_\theta a_\theta^\dagger) + \frac{\mathrm{d}Y_\theta}{\mathrm{d}x}(a_\theta a_\theta - a_\theta^\dagger a_\theta^\dagger)\right] U(x,x_0) \\ &= \left[\frac{1}{2}\frac{\mathrm{d}X_\theta}{\mathrm{d}x} + \frac{\mathrm{d}X_\theta}{\mathrm{d}x} a_\theta^\dagger a_\theta + \frac{1}{2}\frac{\mathrm{d}Y_\theta}{\mathrm{d}x}(a_\theta a_\theta - a_\theta^\dagger a_\theta^\dagger)\right] U(x,x_0). \end{aligned} \tag{4.11}$$

By the same reason as explained with respect to equations (1.26) and (1.27), the term $\frac{1}{2}\frac{\mathrm{d}X_\theta}{\mathrm{d}x}$ in the last expression of (4.11) can be dropped; expression (4.9) is not affected by the omission of this term. So we use, instead of (4.11),

$$\frac{\partial}{\partial x}U(x,x_0) = \left[\frac{\mathrm{d}X_\theta}{\mathrm{d}x} a_\theta^\dagger a_\theta + \frac{1}{2}\frac{\mathrm{d}Y_\theta}{\mathrm{d}x}(a_\theta a_\theta - a_\theta^\dagger a_\theta^\dagger)\right] U(x,x_0). \tag{4.12}$$

The first term in the parentheses on the right-hand side describes the free propagation of the a_θ boson. We can see that the free propagator connecting the points x_1 and x_2 is $\exp[X_\theta(x_2) - X_\theta(x_1)]$. The remaining terms, which describe the pair creation and pair annihilation of a_θ bosons, are the interaction terms. The coupling constant of interaction is $\frac{1}{2}\frac{dY_\theta}{dx}$ for pair creation, and $-\frac{1}{2}\frac{dY_\theta}{dx}$ for pair annihilation.

Let us set $\theta = \pm\pi/2$ in the above expressions. We use the following shorthand notation:

$$\tilde{a}_\pm \equiv a_{\pm\pi/2} \qquad \tilde{a}_\pm^\dagger \equiv a_{\pm\pi/2}^\dagger \qquad \xi_\pm \equiv \xi_{\pm\pi/2} \qquad \xi'_\pm \equiv \xi'_{\pm\pi/2}. \tag{4.13}$$

From (3.14) and (3.15*a*) we find

$$X_{\pm\pi/2}(x) = \pm\tfrac{1}{2}V(x) \qquad Y_{\pm\pi/2}(x) = \mp \mathrm{i}kx \tag{4.14}$$

and so the evolution equation (4.12) with $\theta = \pm\pi/2$ reads

$$\frac{\partial}{\partial x}U(x, x_0) = \frac{1}{2}\left[\pm\frac{\mathrm{d}V(x)}{\mathrm{d}x}\tilde{a}_\pm^\dagger\tilde{a}_\pm \mp \mathrm{i}k(\tilde{a}_\pm\tilde{a}_\pm - \tilde{a}_\pm^\dagger\tilde{a}_\pm^\dagger)\right]U(x, x_0). \tag{4.15}$$

From (4.15) we can see that the coupling constant of interaction is $\mathrm{i}k/2$, apart from a sign. In other words, the expansion in terms of the interaction gives an expansion in terms of k, or $\sqrt{E}$. Thus, by working with $\theta = \pm\pi/2$, we can obtain a low-energy expansion. Setting $\theta = \pm\pi/2$ in (4.9), we have

$$\bar{G} = 2\frac{\langle 0;\pi/2|\mathrm{e}^{\xi_+\tilde{a}_+\tilde{a}_+/2}U(z,x)\tilde{a}_+^\dagger U(x,x')\tilde{a}_+^\dagger U(x',z')\mathrm{e}^{\xi'_-\tilde{a}_+^\dagger\tilde{a}_+^\dagger/2}|0;\pi/2\rangle}{\langle 0;\pi/2|\mathrm{e}^{\xi_+\tilde{a}_+\tilde{a}_+/2}U(z,z')\mathrm{e}^{\xi'_-\tilde{a}_+^\dagger\tilde{a}_+^\dagger/2}|0;\pi/2\rangle} \tag{4.16a}$$

$$= 2\frac{\langle 0;-\pi/2|\mathrm{e}^{\xi_-\tilde{a}_-\tilde{a}_-/2}U(z,x)\tilde{a}_- U(x,x')\tilde{a}_- U(x',z')\mathrm{e}^{\xi'_+\tilde{a}_-^\dagger\tilde{a}_-^\dagger/2}|0;-\pi/2\rangle}{\langle 0;-\pi/2|\mathrm{e}^{\xi_-\tilde{a}_-\tilde{a}_-/2}U(z,z')\mathrm{e}^{\xi'_+\tilde{a}_-^\dagger\tilde{a}_-^\dagger/2}|0;-\pi/2\rangle}. \tag{4.16b}$$

These expressions have almost the same structure as the expression for $\theta = 0$ given by equation (3.1). The only difference is that the operator $a + a^\dagger$ has been replaced by $\sqrt{2}\tilde{a}_+^\dagger$ or $\sqrt{2}\tilde{a}_-$.

In section 1 we derived the boson representation from the representation in terms of paths. Now we may reverse this procedure; the description of the scattering process in terms of the $\tilde{a}_\pm$ boson can be put back to a description in terms of paths. The free propagator of the $\tilde{a}_\pm$ boson, $\mathrm{e}^{\pm[V(x_2)-V(x_1)]/2}$, is represented by a line connecting the points x_2 and x_1. The pair creation or annihilation of $\tilde{a}_\pm$ bosons corresponds to the turning point of the path, and a scattering factor $\pm\mathrm{i}k$ is assigned to each turning point. (Taking account of the symmetry factor, we need to multiply by 2 the coefficient $\frac{1}{2}\mathrm{i}k$ in (4.15).) There is also the scattering described by the operators $\mathrm{e}^{\xi_\pm\tilde{a}_\pm\tilde{a}_\pm/2}$ and $\mathrm{e}^{\xi'_\mp\tilde{a}_\pm^\dagger\tilde{a}_\pm^\dagger/2}$ in (4.16). This gives a factor $\xi_\pm$ and $\xi'_\mp$ at the points z and z', respectively. In summary, we obtain $\bar{G}$ as the sum of diagrams (paths) with the rules shown in figure 6. Since the operator assigned to the endpoints of a path is now $\sqrt{2}\tilde{a}_+^\dagger$ or $\sqrt{2}\tilde{a}_-$, the line at the endpoints sets out to the right for $\theta = +\pi/2$, and to the left for $\theta = -\pi/2$. The two factors $\sqrt{2}$ bring forth an overall factor 2, as in equations (4.16). This diagrammatic expansion gives an expansion in powers of k.

Let us calculate the first few terms of expansion. We work with $\theta = +\pi/2$, and use expression (4.16*a*). (It is easy to see that the same result is obtained with $\theta = -\pi/2$.) The propagator is expanded in powers of k as

$$\bar{G} = g_0 + kg_1 + k^2g_2 + \cdots. \tag{4.17}$$

Figure 7 shows the diagrams contributing to the zeroth-order term g_0. These are the paths that have no turning points except at z and z'. (The rules for interpreting the diagrams are given by

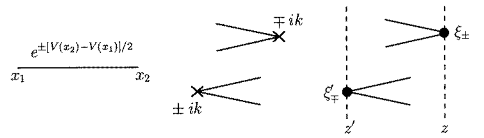

Figure 6. Diagrammatic rules in the frame with angle $\theta = \pm\pi/2$.

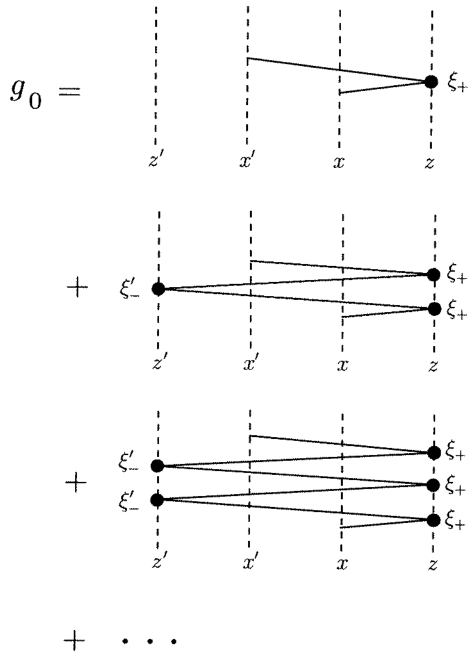

Figure 7. Diagrammatic representation of g_0.

figure 6 with the upper sign. As explained above, there is an overall factor 2.) These diagrams sum up to

$$
\begin{aligned}
g_0 &= 2\xi_+ \mathrm{e}^{[V(z)-V(x)]/2}\mathrm{e}^{[V(z)-V(x')]/2} \\
&\quad \times [1 + \xi_+\xi'_-\mathrm{e}^{V(z)-V(z')} + (\xi_+\xi'_-\mathrm{e}^{V(z)-V(z')})^2 + (\xi_+\xi'_-\mathrm{e}^{V(z)-V(z')})^3 + \cdots] \\
&= \frac{2\xi_+\mathrm{e}^{[2V(z)-V(x)-V(x')]/2}}{1 - \mathrm{e}^{V(z)-V(z')}\xi_+\xi'_-} = \frac{-2\mathrm{e}^{-[V(x)+V(x')]/2}}{\mathrm{e}^{-V(z)}\xi_- + \mathrm{e}^{-V(z')}\xi'_-}
\end{aligned}
\tag{4.18}
$$

where we have used $\xi_+ = -1/\xi_-$. If we define

$$
\hat{\xi}_\pm \equiv \mathrm{e}^{\pm V(z)}\xi_\pm \qquad \hat{\xi}'_\pm \equiv \mathrm{e}^{\pm V(z')}\xi'_\pm \tag{4.19}
$$

then (4.18) becomes

$$
g_0 = 2\mathrm{e}^{-[V(x)+V(x')]/2}\frac{-1}{\hat{\xi}_- + \hat{\xi}'_-}. \tag{4.20}
$$

The terms of order k can be calculated in the same way. The diagrams contributing to g_1 are shown in figure 8. Here we have defined the 'renormalized' propagator as in figure 9, which amounts to

$$
\begin{aligned}
&\mathrm{e}^{[V(x_2)-V(x_1)]/2}[1 + \xi_+\xi'_-\mathrm{e}^{V(z)-V(z')} + (\xi_+\xi'_-\mathrm{e}^{V(z)-V(z')})^2 + \cdots] \\
&\qquad = \frac{\mathrm{e}^{[V(x_2)-V(x_1)]/2}}{1 - \mathrm{e}^{V(z)-V(z')}\xi_+\xi'_-}.
\end{aligned}
\tag{4.21}
$$

Using (4.21), we can calculate the contribution from the upper-left diagram of figure 8 as

$$
-2\mathrm{i}k\int_x^z \left(\frac{\mathrm{e}^{[V(y)-V(x')]/2}}{1 - \mathrm{e}^{V(z)-V(z')}\xi_+\xi'_-}\right)\left(\frac{\mathrm{e}^{[V(y)-V(x)]/2}}{1 - \mathrm{e}^{V(z)-V(z')}\xi_+\xi'_-}\right)\mathrm{d}y
$$

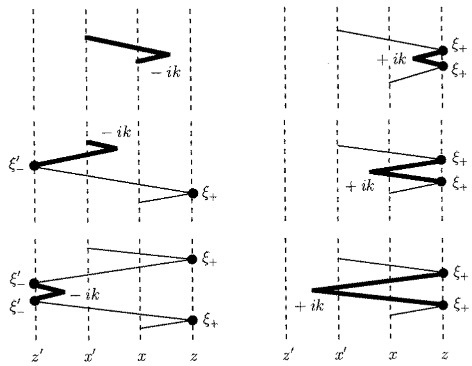

Figure 8. The diagrams contributing to g_1. The heavy line is defined in figure 9.

$$= -2\mathrm{i}k\mathrm{e}^{-[V(x)+V(x')]/2}\left(\frac{1}{\hat{\xi}_- + \hat{\xi}'_-}\right)^2 \hat{\xi}_-^2 \int_x^z \mathrm{e}^{V(y)}\mathrm{d}y \tag{4.22}$$

where the diagram has been integrated over the position of the turning point in the region (x, z). Similarly, the upper-right diagram amounts to

$$2\mathrm{i}k\xi_+^2 \int_x^z \mathrm{e}^{[V(z)-V(x')]/2}\left(\frac{\mathrm{e}^{[V(z)-V(y)]/2}}{1-\mathrm{e}^{V(z)-V(z')}\xi_+\xi'_-}\right)^2 \mathrm{e}^{[V(z)-V(x)]/2}\mathrm{d}y$$
$$= 2\mathrm{i}k\mathrm{e}^{-[V(x)+V(x')]/2}\left(\frac{1}{\hat{\xi}_- + \hat{\xi}'_-}\right)^2 \int_x^z \mathrm{e}^{-V(y)}\mathrm{d}y. \tag{4.23}$$

We can evaluate all the six diagrams of figure 8 in this way. Summing them up we obtain

$$g_1 = 2\mathrm{i}\mathrm{e}^{-[V(x)+V(x')]/2}\left(\frac{1}{\hat{\xi}_- + \hat{\xi}'_-}\right)^2 \times\left[\int_{z'}^z \mathrm{e}^{-V(y)}\mathrm{d}y - (\hat{\xi}'_-)^2\int_{z'}^{x'} \mathrm{e}^{V(y)}\mathrm{d}y + \hat{\xi}_-\hat{\xi}'_-\int_{x'}^{x} \mathrm{e}^{V(y)}\mathrm{d}y - \hat{\xi}_-^2\int_{x}^{z} \mathrm{e}^{V(y)}\mathrm{d}y\right]. \tag{4.24}$$

We can go on in this way and calculate g_2, g_3, etc. However, this procedure becomes more complicated for higher-order terms. Although this representation in terms of paths is instructive, it is not particularly convenient for the practical calculation of these expansion coefficients. To calculate these coefficients in a systematic way, we introduce in the next section a different representation of the scattering process, which makes use of a different kind of boson operators.

5. Tree representation: another boson picture

Figure 10(*a*) shows a typical path contributing to the left reflection coefficient R_l for a certain interval (see equation (2.5*c*)). We may notice that this path can be represented by the 'tree' shown in figure 10(*b*). The correspondence between the path and the tree is illustrated in figure 11. As in this example, every path contributing to R_l can be uniquely represented by a tree, with the rules given by figure 11. This motivates us to introduce another kind of boson

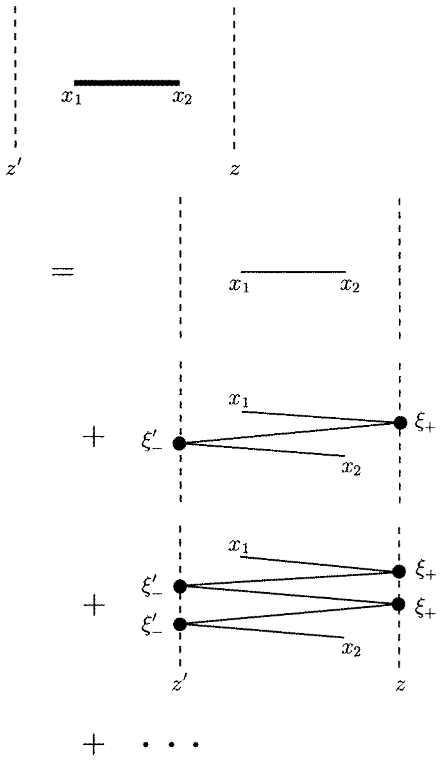

Figure 9. Definition of the 'renormalized' propagator (equation (4.21)).

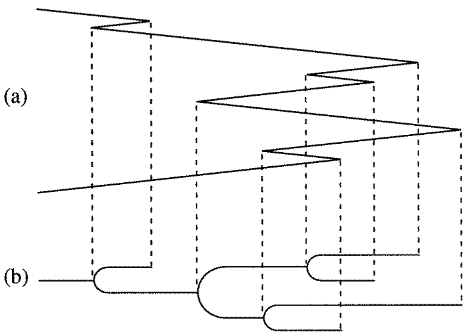

Figure 10. (*a*) A path contributing to the left reflection coefficient R_l. (*b*) The tree corresponding to the above path.

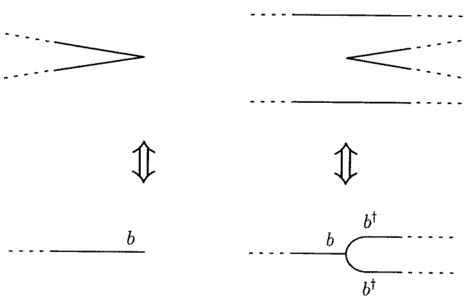

Figure 11. The correspondence between the path and the tree.

operator for describing the scattering process; we interpret the tree in figure 10(*b*) as made up of propagation lines of this boson.

Let b and $b^\dagger$ denote the annihilation and creation operators of this new boson. They satisfy the commutation relation

$$[b, b^\dagger] = 1. \tag{5.1}$$

Recall that the operators $J_\pm$ are represented as $J_+ = -a^\dagger a^\dagger/2$ and $J_- = aa/2$ in terms of

a and $a^\dagger$ (see equations (3.3a)). It can be seen from figure 11 that the pair annihilation of a-bosons corresponds to the annihilation of a single b-boson, and that the pair creation of a-bosons corresponds to the splitting of a b-boson in two. So we can guess that it would be possible to represent J_+ and J_- by the operators $-b^\dagger b^\dagger b$ and b, respectively. The operator J_3 should describe the free propagation of the b-boson, and so it would have the form $b^\dagger b$. Indeed, if we write

$$J_+ = -b^\dagger b^\dagger b \qquad J_- = b \qquad J_3 = b^\dagger b \tag{5.2}$$

we can easily see that these operators satisfy the commutation relations (3.5a), provided that b and $b^\dagger$ satisfy (5.1).

However, (5.2) is not sufficient for describing general scattering processes. Let us consider the diagram shown in figure 12(a). This is a typical path contributing to the transmission coefficient τ for a certain interval (equation (2.5a)). Such a path cannot be represented by a single tree as in figure 10(b). Instead, it can be represented by a group of trees, as shown in figure 12(b). From figure 12 it is obvious that the operator J_+ must now include the creation term of a single b-boson. Namely, the first equation of (5.2) should be replaced by $J_+ = -b^\dagger b^\dagger b - b^\dagger$. In accordance with this, the operator J_3 should be modified to $J_3 = b^\dagger b + \frac{1}{2}$, in order to satisfy the commutation relations (3.5a). Thus, to represent the scattering process corresponding to the path in figure 12(a), we have to adopt

$$J_+ = -b^\dagger b^\dagger b - b^\dagger \qquad J_- = b \qquad J_3 = b^\dagger b + \tfrac{1}{2}. \tag{5.3}$$

In order to give a general description of scattering processes, (5.2) and (5.3) must be unified. That is to say, we must find a generalized expression that includes both (5.2) and (5.3) as special cases. This can be achieved by introducing fermion operators. Let c and $c^\dagger$ be fermion annihilation and creation operators, satisfying

$$c^2 = (c^\dagger)^2 = 0 \qquad \{c, c^\dagger\} = cc^\dagger + c^\dagger c = 1. \tag{5.4}$$

(These operators commute with the boson operators b and $b^\dagger$.) Using these fermion operators, we can put (5.2) and (5.3) together as

$$J_+ = -b^\dagger b^\dagger b - b^\dagger c^\dagger c \qquad J_- = b \qquad J_3 = b^\dagger b + \tfrac{1}{2} c^\dagger c. \tag{5.5}$$

The expressions (5.2) and (5.3) are recovered by restricting (5.5) to eigenstates of the fermion number operator $c^\dagger c$, with eigenvalue 0 and 1, respectively. Now the creation term of a single b-boson in J_+ (the first equation of (5.3)) is re-interpreted as the emission of a b-boson from a c-fermion. The path in figure 12(a) is represented by a tree as in figure 12(c), with a fermion line as the trunk.

It is rather trivial to see that (5.5) satisfies (3.5a). The real merit of introducing the fermion is that it enables us to construct a representation of the superalgebra (3.4). As we saw before, Q_+ and Q_- describe the endpoints of a path. From the correspondence between the path and the tree, it is obvious that $Q_\pm$ should include either creation or annihilation of a fermion. So they would be expressed as $Q_+ = A_1 c + A_2 c^\dagger$ and $Q_- = A_3 c + A_4 c^\dagger$, where A_i ($i = 1, 2, 3, 4$) are functions of the operators b and $b^\dagger$. It is not difficult to find A_i such that Q_+ and Q_- satisfy the relations (3.4) together with the operators (5.5). The answer is

$$Q_+ = b^\dagger c + b^\dagger b c^\dagger \qquad Q_- = c + b c^\dagger. \tag{5.6}$$

We can easily check that (5.5) and (5.6) satisfy (3.4).

There is another way of representing a path by a tree. For example, look at figure 13(a). This is a typical path contributing to the right reflection coefficient R_r (equation (2.5b)). It is more natural to represent such a path by a tree that grows in the opposite direction, as in figure 13(b). In other words, it is more natural to use

$$J_+ = -b^\dagger \qquad J_- = b^\dagger b b + b c^\dagger c \qquad J_3 = b^\dagger b + \tfrac{1}{2} c^\dagger c \tag{5.7}$$

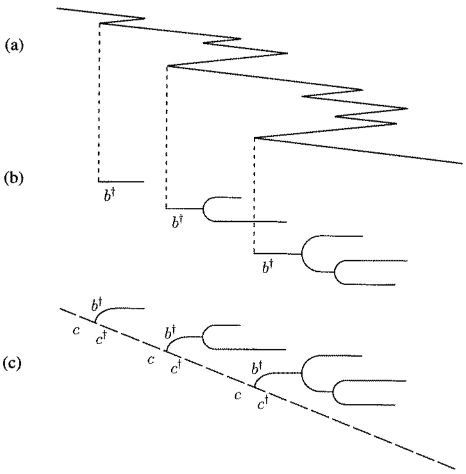

Figure 12. (*a*) A path contributing to the transmission coefficient τ. (*b*) Tree representation of the above path. (*c*) Re-interpretation of (*b*). The broken line is the propagation line of the fermion.

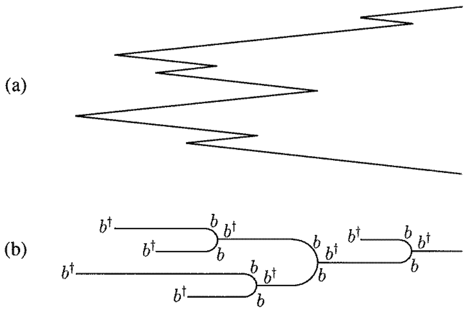

Figure 13. (*a*) A path contributing to the right reflection coefficient R_r. (*b*) Representation of the above path in terms of a tree that grows to the left.

rather than (5.5). The operators $Q_\pm$ that satisfy (3.4) with (5.7) can be found as

$$Q_+ = c^\dagger + b^\dagger c \qquad Q_- = bc^\dagger + b^\dagger bc. \tag{5.8}$$

Thus, we have two different ways of realizing the superalgebra (3.4) in terms of b, $b^\dagger$, c, and $c^\dagger$. We use the superscripts L and R to distinguish one from the other; we write

$$\begin{array}{lll} J_+^L = -b^\dagger b^\dagger b - b^\dagger c^\dagger c & J_-^L = b & J_3^L = b^\dagger b + \frac{1}{2}c^\dagger c \\ Q_+^L = b^\dagger c + b^\dagger bc^\dagger & Q_-^L = c + bc^\dagger & \end{array} \tag{5.9a}$$

and

$$\begin{array}{lll} J_+^R = -b^\dagger & J_-^R = b^\dagger bb + bc^\dagger c & J_3^R = b^\dagger b + \frac{1}{2}c^\dagger c \\ Q_+^R = c^\dagger + b^\dagger c & Q_-^R = bc^\dagger + b^\dagger bc. & \end{array} \tag{5.9b}$$

Both $\{J_3^L, J_\pm^L, Q_\pm^L\}$ and $\{J_3^R, J_\pm^R, Q_\pm^R\}$ satisfy relations (3.4).

6. Propagator in the tree representation

Let us define $|0\rangle$ to be the vacuum state satisfying $b|0\rangle = 0$ and $c|0\rangle = 0$. (Although we are using the same notation as the vacuum of the a-boson, there would be no danger of confusion.) The Fock space generated by applying $b^\dagger$ and $c^\dagger$ to $|0\rangle$ is the representation space on which the operators (5.9) act. We define the bra conjugate to a ket as usual. In particular, $\langle 0|$ satisfies $\langle 0|b^\dagger = 0$ and $\langle 0|c^\dagger = 0$. Using the bra and the ket, the ordinary inner product in the Fock space is expressed as $\langle \Psi|\Phi\rangle$. We assume that the vacuum state is normalized as $\langle 0|0\rangle = 1$.

We wish to write the expression (3.11) in our new representation (5.9). To do so, we have to define an inner product (Ψ, Φ). Now it is not appropriate to adopt $(\Psi, \Phi) = \langle \Psi|\Phi\rangle$, since this inner product does not satisfy the requirement (3.8) with either $Q_\pm^L$ or $Q_\pm^R$.

Let us notice that the operators J^R, Q^R and J^L, Q^L are related by

$$MQ_\pm^R = Q_\pm^L M \qquad MJ_a^R = J_a^L M \qquad (a = 1, 2, 3, \pm) \tag{6.1}$$

where

$$M \equiv \Gamma(b^\dagger b + c^\dagger c). \tag{6.2}$$

Here Γ denotes the gamma function. This operator M is diagonal in the number states. Definition (6.2) means

$$M(c^\dagger)^m(b^\dagger)^n|0\rangle = (n+m-1)!(c^\dagger)^m(b^\dagger)^n|0\rangle \tag{6.3}$$

and therefore

$$\langle 0|b^{n'}c^{m'}M(c^\dagger)^m(b^\dagger)^n|0\rangle = (n+m-1)!n!\delta_{mm'}\delta_{nn'} \qquad (m=0 \text{ or } 1). \tag{6.4}$$

Relations (6.1) can be readily verified by comparing the matrix elements of both sides in the number states.

Using the operator M, we define

$$(\Psi,\Phi) \equiv \langle\Psi|M|\Phi\rangle. \tag{6.5}$$

From (5.9) we find that $Q^L_\pm$ is the adjoint of $Q^R_\mp$ in the usual sense (that is, adjoint with respect to the ordinary inner product in the Fock space). Hence we can see that the inner product (6.5) satisfies (3.8). Indeed, we have

$$(\Psi, Q^R_\pm\Phi) = \langle\Psi|MQ^R_\pm|\Phi\rangle = \langle\Psi|Q^L_\pm M|\Phi\rangle = (Q^R_\mp\Psi,\Phi). \tag{6.6}$$

As can be seen from (6.6), with this inner product the operators $Q^R_\pm$ play the the role of $Q_\pm$ in (3.8).

There is a subtle point here. Since the gamma function has a pole at the origin, the operator M produces an infinity when applied to the vacuum state. In other words, equation (6.3) is not well defined for $n=m=0$. This difficulty can be circumvented as follows. We remark that (5.9) can be generalized to the form

$$\begin{aligned} &J^L_+ = -b^\dagger b^\dagger b - b^\dagger(c^\dagger c+\nu) \qquad J^L_- = b \qquad J^L_3 = b^\dagger b + \tfrac{1}{2}(c^\dagger c+\nu) \\ &Q^L_+ = b^\dagger c + (b^\dagger b+\nu)c^\dagger \qquad Q^L_- = c + bc^\dagger \end{aligned} \tag{6.7a}$$

and

$$\begin{aligned} &J^R_+ = -b^\dagger \qquad J^R_- = b^\dagger bb + b(c^\dagger c+\nu) \qquad J^R_3 = b^\dagger b + \tfrac{1}{2}(c^\dagger c+\nu) \\ &Q^R_+ = c^\dagger + b^\dagger c \qquad Q^R_- = bc^\dagger + (b^\dagger b+\nu)c \end{aligned} \tag{6.7b}$$

where ν is an arbitrary parameter. We can easily see that these operators still satisfy (3.4) for arbitrary ν. Relations (6.1) remain satisfied if we modify the operator M to

$$M = \frac{\Gamma(b^\dagger b + c^\dagger c + \nu)}{\Gamma(\nu+1)}. \tag{6.8}$$

Using the formula $\Gamma(x)/\Gamma(x+1) = 1/x$, we find

$$M|0\rangle = \frac{1}{\nu}|0\rangle. \tag{6.9}$$

Whenever $M|0\rangle$ or $\langle 0|M$ appears in calculation, we should use (6.7) and (6.8) with finite ν, and then take the limit $\nu\to 0$. For example, although the expression $J^L_3 M|0\rangle$ is not well defined with (5.9) and (6.2), we can calculate it with (6.7) and (6.8) as

$$J^L_3 M|0\rangle = \frac{1}{\nu}J^L_3|0\rangle = \frac{1}{\nu}\frac{\nu}{2}|0\rangle = \frac{1}{2}|0\rangle. \tag{6.10}$$

Unless the ill-defined expressions appear, we can use (5.9) and (6.2) rather than (6.7) and (6.8). (As a matter of fact, we do not need (6.7) and (6.8) any more in this paper, now that we have derived (6.10).)

Writing (3.12) with the specific operators J^L_a and J^R_a given by (5.9), we define the evolution operators U^L and U^R by

$$\frac{\partial}{\partial x}U^{L,R}(x,x_0) = [2\mathrm{i}kJ^{L,R}_3 - 2f(x)J^{L,R}_1]U^{L,R}(x,x_0) \tag{6.11}$$

with the initial condition $U^{L,R}(x = x_0) = 1$. From (6.1) it is obvious that

$$MU^R = U^L M. \tag{6.12}$$

Now we are ready to write (3.11) in our present representation. The lowest state Ψ_0 satisfying $Q_-^R \Psi_0 = 0$ is the vacuum state $|0\rangle$. Using the inner product (6.5), we can write

$$\begin{aligned}\bar{G} &= \frac{\langle 0|M\mathrm{e}^{\xi J_-^R} U^R(z,x)(Q_+^R + Q_-^R)U^R(x,x')(Q_+^R + Q_-^R)U^R(x',z')\mathrm{e}^{-\xi' J_+^R}|0\rangle}{2\langle 0|M\mathrm{e}^{\xi J_-^R} U^R(z,z')\mathrm{e}^{-\xi' J_+^R} J_3^R|0\rangle} \\ &= \frac{\langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,x)(Q_+^L + Q_-^L)U^L(x,x')(Q_+^L + Q_-^L)U^L(x',z')^{-\xi' J_+^L} M|0\rangle}{2\langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,z')\mathrm{e}^{-\xi' J_+^L} J_3^L M|0\rangle}\end{aligned} \tag{6.13}$$

where we have used (6.1) and (6.12) to derive the second line. The quantity in the denominators of the expressions (6.13) turns out to be unity, as we shall now see. From (5.9*a*) and $b|0\rangle = c|0\rangle = 0$ it follows that

$$J_\pm^L|0\rangle = 0 \qquad J_3^L|0\rangle = 0. \tag{6.14}$$

In view of (6.14) and (6.11), it is obvious that U^L does not affect $|0\rangle$:

$$U^L|0\rangle = |0\rangle. \tag{6.15}$$

So, using (6.10), (6.14) and (6.15) we find

$$\begin{aligned}2\langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,z')\mathrm{e}^{-\xi' J_+^L} J_3^L M|0\rangle &= \langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,z')\mathrm{e}^{-\xi' J_+^L}|0\rangle \\ &= \langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,z')|0\rangle = \langle 0|\mathrm{e}^{\xi J_-^L}|0\rangle = \langle 0|0\rangle = 1.\end{aligned} \tag{6.16}$$

Consequently, (6.13) takes the simple form

$$\begin{aligned}\bar{G} &= \langle 0|M\mathrm{e}^{\xi J_-^R} U^R(z,x)(Q_+^R + Q_-^R)U^R(x,x')(Q_+^R + Q_-^R)U^R(x',z')\mathrm{e}^{-\xi' J_+^R}|0\rangle \\ &= \langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,x)(Q_+^L + Q_-^L)U^L(x,x')(Q_+^L + Q_-^L)U^L(x',z')\mathrm{e}^{-\xi' J_+^L} M|0\rangle.\end{aligned} \tag{6.17}$$

By using (6.1) and (6.12), we may move the operator M to anywhere in the middle of the expression; for example,

$$\begin{aligned}\bar{G} &= \langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,x)(Q_+^L + Q_-^L)MU^R(x,x')(Q_+^R + Q_-^R)U^R(x',z')\mathrm{e}^{-\xi' J_+^R}|0\rangle \\ &= \langle 0|\mathrm{e}^{\xi J_-^L} U^L(z,x)(Q_+^L + Q_-^L)U^L(x,x')M(Q_+^R + Q_-^R)U^R(x',z')\mathrm{e}^{-\xi' J_+^R}|0\rangle.\end{aligned} \tag{6.18}$$

The expressions (6.18) are more convenient than (6.17) as they do not include $M|0\rangle$ or $\langle 0|M$, which requires a special care. We shall use the last expression of (6.18) for our further analysis.

We define

$$|\eta'\rangle \equiv \mathrm{e}^{\eta' b^\dagger}|0\rangle \qquad \langle\eta| \equiv \langle 0|\mathrm{e}^{\eta b} \tag{6.19}$$

where η and η' are arbitrary complex numbers. (It would be more appropriate to write $\langle\eta^*|$ instead of $\langle\eta|$ here, since it is the bra conjugate to the ket $|\eta^*\rangle$. However, we omit the asterisk just for simplicity.) The states defined by (6.19) are the coherent states, which have the properties

$$b|\eta'\rangle = \eta'|\eta'\rangle \qquad \langle\eta|b^\dagger = \eta\langle\eta| \qquad b^\dagger|\eta'\rangle = \frac{\partial}{\partial\eta'}|\eta'\rangle \qquad \langle\eta|b = \frac{\partial}{\partial\eta}\langle\eta|. \tag{6.20}$$

Since $J_-^L = b$ and $J_+^R = -b^\dagger$, we can write (6.18) in terms of the coherent states as

$$\bar{G} = \langle\xi|U^L(z,x)(Q_+^L + Q_-^L)U^L(x,x')M(Q_+^R + Q_-^R)U^R(x',z')|\xi'\rangle. \tag{6.21}$$

Let us also remark that the scattering coefficients have simple forms in this representation. From (3.2), (6.18)and (5.9) we have

$$\begin{aligned}\tau(x_2, x_1) &= \langle 0|cU^R(x_2, x_1)c^\dagger|0\rangle = \langle 0|cU^L(x_2, x_1)c^\dagger|0\rangle \\ R_r(x_2, x_1) &= \langle 0|bU^R(x_2, x_1)|0\rangle \qquad R_l(x_2, x_1) = \langle 0|U^L(x_2, x_1)b^\dagger|0\rangle.\end{aligned} \tag{6.22}$$

These expressions indeed agree with the graphical interpretation shown in figures 12(*c*), 13(*b*) and 10(*b*).

7. Low-energy expansion of the propagator

This new boson representation turns out to be extremely useful in studying the low-energy behaviour of the propagator. The expansion coefficients g_i of (4.17) can be systematically calculated by using this representation, as we show in this section.

Let us consider the rotation introduced in section 3. The rotation operator is given by the general expression (3.19). Since we are now dealing with two sets of operators J_a (namely, J_a^L and J_a^R), we need to consider the corresponding two rotation operators $P^L(\theta)$ and $P^R(\theta)$ defined by

$$P^{L,R}(\theta) = \mathrm{e}^{\tan(\theta/2)J_-^{L,R}}[\cos(\theta/2)]^{2J_3^{L,R}}\mathrm{e}^{-\tan(\theta/2)J_+^{L,R}}. \tag{7.1}$$

Correspondingly, we define the two sets of boson and fermion operators $\{b_\theta^L, b_\theta^{\dagger L}, c_\theta^L, c_\theta^{\dagger L}\}$ and $\{b_\theta^R, b_\theta^{\dagger R}, c_\theta^R, c_\theta^{\dagger R}\}$ as

$$\begin{aligned} b_\theta^{L,R} &\equiv P^{L,R}(\theta)bP^{L,R}(-\theta) \qquad b_\theta^{\dagger L,R} \equiv P^{L,R}(\theta)b^\dagger P^{L,R}(-\theta) \\ c_\theta^{L,R} &\equiv P^{L,R}(\theta)cP^{L,R}(-\theta) \qquad c_\theta^{\dagger L,R} \equiv P^{L,R}(\theta)c^\dagger P^{L,R}(-\theta).\end{aligned} \tag{7.2}$$

We also have two sets of J and Q operators with angle θ,

$$J_{a,\theta}^{L,R} \equiv P^{L,R}(\theta)J_a^{L,R}P^{L,R}(-\theta) \qquad Q_{\pm,\theta}^{L,R} \equiv P^{L,R}(\theta)Q_\pm^{L,R}P^{L,R}(-\theta). \tag{7.3}$$

The operators (7.3) satisfy the same relations as (3.15*b*) and (3.26):

$$\begin{pmatrix} J_{3,\theta}^{L,R} \\ J_{1,\theta}^{L,R}\end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} J_3^{L,R} \\ J_1^{L,R}\end{pmatrix} \tag{7.4a}$$

$$\begin{pmatrix} Q_{+,\theta}^{L,R} \\ Q_{-,\theta}^{L,R}\end{pmatrix} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2)\end{pmatrix}\begin{pmatrix} Q_+^{L,R} \\ Q_-^{L,R}\end{pmatrix} \tag{7.4b}$$

The boson and fermion operators (7.2) have more complicated forms. As shown in appendix D, we have

$$\begin{aligned} b_\theta^L &= [\cos(\theta/2) - \sin(\theta/2)b^\dagger]^2 b - \sin(\theta/2)[\cos(\theta/2) - \sin(\theta/2)b^\dagger]c^\dagger c \\ b_\theta^{\dagger L} &= \frac{\sin(\theta/2) + \cos(\theta/2)b^\dagger}{\cos(\theta/2) - \sin(\theta/2)b^\dagger} \\ c_\theta^L &= [\cos(\theta/2) - \sin(\theta/2)b^\dagger]c \qquad c_\theta^{\dagger L} = \frac{c^\dagger}{\cos(\theta/2) - \sin(\theta/2)b^\dagger}\end{aligned} \tag{7.5a}$$

and

$$\begin{aligned} b_\theta^R &= \frac{-\sin(\theta/2) + \cos(\theta/2)b}{\cos(\theta/2) + \sin(\theta/2)b} \\ b_\theta^{\dagger R} &= b^\dagger[\cos(\theta/2) + \sin(\theta/2)b]^2 + \sin(\theta/2)[\cos(\theta/2) + \sin(\theta/2)b]c^\dagger c \\ c_\theta^R &= \frac{c}{\cos(\theta/2) + \sin(\theta/2)b} \qquad c_\theta^{\dagger R} = [\cos(\theta/2) + \sin(\theta/2)b]c^\dagger.\end{aligned} \tag{7.5b}$$

This is a highly nonlinear transformation, in contrast to the simple relation (4.3) for the a-boson. Here the fraction of operators is to be understood as an infinite series; for example,

$$\frac{1}{\cos(\theta/2) - \sin(\theta/2)b^\dagger} = \frac{1}{\cos(\theta/2)} \sum_{n=0}^{\infty} [\tan(\theta/2)b^\dagger]^n. \tag{7.6}$$

We define the vacuum ket and bra in the rotated frame as

$$|0;\theta\rangle \equiv P^R(\theta)|0\rangle \qquad \langle 0;\theta| \equiv \langle 0|P^L(-\theta). \tag{7.7}$$

They satisfy

$$b^R_\theta|0;\theta\rangle = c^R_\theta|0;\theta\rangle = 0 \qquad \langle 0;\theta|b^{\dagger L}_\theta = \langle 0;\theta|c^{\dagger L}_\theta = 0. \tag{7.8}$$

Corresponding to (6.19), we define

$$|\eta';\theta\rangle \equiv e^{\eta' b^{\dagger R}_\theta}|0;\theta\rangle \qquad \langle \eta;\theta| \equiv |0;\theta|e^{\eta b^L_\theta}. \tag{7.9}$$

These states are the coherent states in the frame with angle θ. In fact, they are also coherent states in the original unrotated frame; we can show that, for arbitrary θ,

$$e^{\xi' b^\dagger}|0\rangle = e^{\xi'_{-\theta} b^{\dagger R}_\theta}|0;\theta\rangle \qquad \langle 0|e^{\xi b} = \langle 0;\theta|e^{\xi_\theta b^L_\theta} \tag{7.10}$$

where ξ_θ and $\xi'_{-\theta}$ are defined by (4.8). The derivation of (7.10) is the same as that of (4.7), and it is given in appendix C. With the definitions (6.19) and (7.9), equations (7.10) read

$$|\xi'\rangle = |\xi'_{-\theta};\theta\rangle \qquad \langle\xi| = \langle\xi_\theta;\theta|. \tag{7.11}$$

Thus we find that a coherent state in one frame is also a coherent state in any other frame. Using (7.11), we can rewrite (6.21) as

$$\bar{G} = \langle\xi_\theta;\theta|U^L(z,x)(Q^L_+ + Q^L_-)U^L(x,x')M(Q^R_+ + Q^R_-)U^R(x',z')|\xi'_{-\theta};\theta\rangle \tag{7.12}$$

or, more generally,

$$\bar{G} = \langle\xi_\theta;\theta|U^L(z,x)(Q^L_+ + Q^L_-)U^L(x,x')M(Q^R_+ + Q^R_-)U^R(x',z')|\xi'_{-\theta'};\theta'\rangle \tag{7.13}$$

where both θ and θ' are arbitrary.

As in section 4, we can obtain the expansion in terms of k by working in the frame with angle $\pm\pi/2$. Before going on, let us make some definitions in order to simplify the notation. We write

$$\tilde{b}^L \equiv b^L_{-\pi/2} \qquad \tilde{b}^{\dagger L} \equiv b^{\dagger L}_{-\pi/2} \qquad \tilde{c}^L \equiv c^L_{-\pi/2} \qquad \tilde{c}^{\dagger L} \equiv c^{\dagger L}_{-\pi/2} \tag{7.14a}$$

$$\tilde{b}^R \equiv b^R_{+\pi/2} \qquad \tilde{b}^{\dagger R} \equiv b^{\dagger R}_{+\pi/2} \qquad \tilde{c}^R \equiv c^R_{+\pi/2} \qquad \tilde{c}^{\dagger R} \equiv c^{\dagger R}_{+\pi/2}. \tag{7.14b}$$

It is obvious that transformation (7.2) preserves the boson and fermion commutation relations. So we have

$$[\tilde{b}^L, \tilde{b}^{\dagger L}] = 1 \qquad [\tilde{b}^R, \tilde{b}^{\dagger R}] = 1 \tag{7.15a}$$

$$\{\tilde{c}^L, \tilde{c}^{\dagger L}\} = 1 \qquad \{\tilde{c}^R, \tilde{c}^{\dagger R}\} = 1 \tag{7.15b}$$

$$(\tilde{c}^L)^2 = (\tilde{c}^{\dagger L})^2 = 0 \qquad (\tilde{c}^R)^2 = (\tilde{c}^{\dagger R})^2 = 0. \tag{7.15c}$$

The operators $\tilde{b}^L$ and $\tilde{b}^{\dagger L}$ commute with $\tilde{c}^L$ and $\tilde{c}^{\dagger L}$. Similarly, $\tilde{b}^R$ and $\tilde{b}^{\dagger R}$ commute with $\tilde{c}^R$ and $\tilde{c}^{\dagger R}$.

We also use the notation

$$\tilde{J}^L_\pm \equiv J^L_{\pm,-\pi/2} \qquad \tilde{J}^L_3 \equiv J^L_{3,-\pi/2} \qquad \tilde{Q}^L_\pm \equiv Q^L_{\pm,-\pi/2} \tag{7.16a}$$

$$\tilde{J}^R_\pm \equiv -J^R_{\mp,+\pi/2} \qquad \tilde{J}^R_3 \equiv -J^R_{3,+\pi/2} \qquad \tilde{Q}^R_\pm \equiv \mp Q^R_{\mp,+\pi/2}. \tag{7.16b}$$

These operators are expressed in terms of the boson and fermion operators (7.14) as

$$\tilde{J}^L_+ = -\tilde{b}^{\dagger L}\tilde{b}^{\dagger L}\tilde{b}^L - \tilde{b}^{\dagger L}\tilde{c}^{\dagger L}\tilde{c}^L \qquad \tilde{J}^L_- = \tilde{b}^L \qquad \tilde{J}^L_3 = \tilde{b}^{\dagger L}\tilde{b}^L + \tfrac{1}{2}\tilde{c}^{\dagger L}\tilde{c}^L$$
$$\tilde{Q}^L_+ = \tilde{b}^{\dagger L}\tilde{c}^L + \tilde{b}^{\dagger L}\tilde{b}^L\tilde{c}^{\dagger L} \qquad \tilde{Q}^L_- = \tilde{c}^L + \tilde{b}^L\tilde{c}^{\dagger L} \tag{7.17a}$$

and

$$\tilde{J}^R_+ = -\tilde{b}^{\dagger R}\tilde{b}^R\tilde{b}^R - \tilde{b}^R\tilde{c}^{\dagger R}\tilde{c}^R \qquad \tilde{J}^R_- = \tilde{b}^{\dagger R} \qquad \tilde{J}^R_3 = -\tilde{b}^{\dagger R}\tilde{b}^R - \tfrac{1}{2}\tilde{c}^{\dagger R}\tilde{c}^R$$
$$\tilde{Q}^R_+ = -\tilde{b}^R\tilde{c}^{\dagger R} - \tilde{b}^{\dagger R}\tilde{b}^R\tilde{c}^R \qquad \tilde{Q}^R_- = \tilde{c}^{\dagger R} + \tilde{b}^{\dagger R}\tilde{c}^R. \tag{7.17b}$$

Equations (7.17*a*) have exactly the same form as (5.9*a*), but equations (7.17*b*) are different from (5.9*b*) on account of the definition of signs in (7.16*b*). We have defined $\tilde{J}^R$ and $\tilde{Q}^R$ this way, so that $\tilde{J}^L$, $\tilde{Q}^L$ and $\tilde{J}^R$, $\tilde{Q}^R$ may satisfy

$$M\tilde{Q}^R_\pm = \tilde{Q}^L_\pm M \qquad M\tilde{J}^R_a = \tilde{J}^L_a M \tag{7.18}$$

just like (6.1). (The relations (7.18) can be verified using (6.1), (7.4) and (7.16).) Obviously, $\tilde{J}^L_3$, $\tilde{J}^L_\pm$, and $\tilde{Q}^L_\pm$ satisfy the same relations as (3.4). It is easy to see that $\tilde{J}^R_3$, $\tilde{J}^R_\pm$, and $\tilde{Q}^R_\pm$, too, satisfy (3.4); namely, the relations (3.4) hold with J, Q replaced by either $\tilde{J}^L$, $\tilde{Q}^L$ or $\tilde{J}^R$, $\tilde{Q}^R$.

Let us also define

$$|\eta'; R\rangle \equiv |\eta'; +\pi/2\rangle \qquad \langle\eta; L| \equiv \langle\eta; -\pi/2| \tag{7.19}$$

where the right-hand sides are the coherent states (7.9) with $\theta = \pm\pi/2$. From the definition of the coherent states it is obvious that

$$\tilde{b}^R|\eta'; R\rangle = \eta'|\eta'; R\rangle \qquad \langle\eta; L|\tilde{b}^{\dagger L} = \eta\langle\eta; L| \tag{7.20a}$$

$$\tilde{b}^{\dagger R}|\eta'; R\rangle = \frac{\partial}{\partial\eta'}|\eta'; R\rangle \qquad \langle\eta; L|\tilde{b}^L = \frac{\partial}{\partial\eta}\langle\eta; L|. \tag{7.20b}$$

Note that these states also satisfy

$$\tilde{c}^R|\eta'; R\rangle = 0 \qquad \langle\eta; L|\tilde{c}^{\dagger L} = 0. \tag{7.21}$$

From (7.11) and (4.8) it follows that

$$|\eta'; R\rangle = |\zeta'\rangle \qquad \langle\eta; L| = \langle\zeta| \tag{7.22}$$

where $\zeta \equiv (1+\eta)/(1-\eta)$, $\zeta' \equiv (1+\eta')/(1-\eta')$.

Now let us return to (7.13). To obtain an expansion in terms of k, we set $\theta = -\pi/2$ and $\theta' = +\pi/2$. (This choice of signs makes the resulting expression most simple.) Setting $\theta = -\pi/2$ and $\theta' = +\pi/2$ in equation (7.13), and using the relation

$$Q^{L,R}_+ + Q^{L,R}_- = \sqrt{2}\tilde{Q}^{L,R}_- \tag{7.23}$$

which follows from (7.4*b*), we have

$$\bar{G} = 2\langle\xi_-; L|U^L(z,x)\tilde{Q}^L_- U^L(x,x')M\tilde{Q}^R_- U^R(x',z')|\xi'_-; R\rangle. \tag{7.24}$$

(Recall definition (4.13) for ξ_- and ξ'_-.)

The evolution operators U^L and U^R satisfy equation (3.16) (with U and J replaced by $U^{L,R}$ and $J^{L,R}$) with arbitrary θ. We use $\theta = -\pi/2$ for U^L, and $\theta = +\pi/2$ for U^R. Substituting (4.14) into (3.16), and using the notation (7.16), we have

$$\frac{\partial}{\partial x}U^{L,R}(x,x_0) = \left[-\frac{\mathrm{d}V}{\mathrm{d}x}\tilde{J}^{L,R}_3 + \mathrm{i}k(\tilde{J}^{L,R}_+ + \tilde{J}^{L,R}_-)\right]U^{L,R}(x,x_0). \tag{7.25}$$

The term $-\frac{\mathrm{d}V}{\mathrm{d}x}\tilde{J}^{L,R}_3$ in (7.25) describes the free propagation of the $\tilde{b}^{L,R}$ boson and the $\tilde{c}^{L,R}$ fermion. Since $\tilde{J}^L_3 = \tilde{b}^{\dagger L}\tilde{b}^L + \frac{1}{2}\tilde{c}^{\dagger L}\tilde{c}^L$, we can see that the free propagators of the $\tilde{b}^L$ boson and the $\tilde{c}^L$ fermion connecting the points x_1 and x_2 have the values, respectively, $\mathrm{e}^{-[V(x_2)-V(x_1)]}$

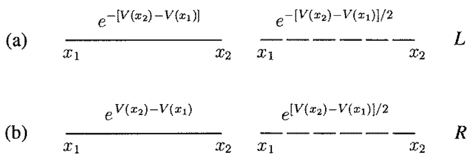

Figure 14. Diagrammatic representation of propagators of: (*a*) $\tilde{b}^L$ boson and $\tilde{c}^L$ fermion; (*b*) $\tilde{b}^R$ boson and $\tilde{c}^R$ fermion.

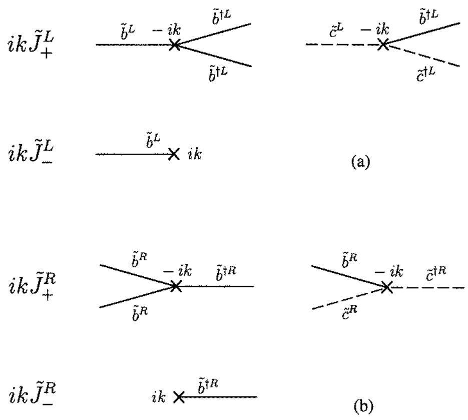

Figure 15. The interaction vertices: (*a*) $\mathrm{i}k\tilde{J}_+^L$ and $\mathrm{i}k\tilde{J}_-^L$, (*b*) $\mathrm{i}k\tilde{J}_+^R$ and $\mathrm{i}k\tilde{J}_-^R$.

and $\mathrm{e}^{-[V(x_2)-V(x_1)]/2}$ (figure 14(*a*)). Similarly, the free propagators of the $\tilde{b}^R$ boson and the $\tilde{c}^R$ fermion are $\mathrm{e}^{V(x_2)-V(x_1)}$ and $\mathrm{e}^{[V(x_2)-V(x_1)]/2}$, respectively (figure 14(*b*)). The operators $\tilde{J}_\pm^{L,R}$ are treated as the interaction, with the coupling constant $\mathrm{i}k$. The interaction vertices, which can be read off from (7.17), are shown in figure 15.

Now we move to the interaction picture, as explained in section 3. The evolution operators in the interaction picture are defined by (3.20). We write

$$\tilde{U}_I^L(x_2,x_1) \equiv U_I^L(x_2,x_1,-\pi/2) = \mathrm{e}^{V(x_2)\tilde{J}_3^L} U^L(x_2,x_1)\mathrm{e}^{-V(x_1)\tilde{J}_3^L} \qquad (7.26a)$$

$$\tilde{U}_I^R(x_2,x_1) \equiv U_I^R(x_2,x_1,+\pi/2) = \mathrm{e}^{V(x_2)\tilde{J}_3^R} U^R(x_2,x_1)\mathrm{e}^{-V(x_1)\tilde{J}_3^R}. \qquad (7.26b)$$

Setting $\theta = \pm\pi/2$ in (3.23), and substituting (4.14), we obtain

$$\frac{\partial}{\partial x}\tilde{U}_I^{L,R}(x,x_0) = \mathrm{i}k(\mathrm{e}^{V(x)}\tilde{J}_+^{L,R} + \mathrm{e}^{-V(x)}\tilde{J}_-^{L,R})\tilde{U}_I^{L,R}(x,x_0). \qquad (7.27)$$

This differential equation, with the initial condition $\tilde{U}_I^{L,R}(x=x_0)=1$, is equivalent to the integral equation

$$\tilde{U}_I^{L,R}(x,x_0) = 1 + \mathrm{i}k\int_{x_0}^{x}\mathrm{d}y(\mathrm{e}^{V(y)}\tilde{J}_+^{L,R} + \mathrm{e}^{-V(y)}\tilde{J}_-^{L,R})\tilde{U}_I^{L,R}(y,x_0). \qquad (7.28)$$

From (7.28) we can obtain the expansion of $\tilde{U}_I^{L,R}$ in powers of k. Let us introduce the notation

$$[s_1,s_2,\ldots,s_n]_{x_1}^{x_2} \equiv \int\cdots\int_{x_1\leqslant y_1\leqslant y_2\leqslant\cdots\leqslant y_n\leqslant x_2}\mathrm{d}y_1\cdots\mathrm{d}y_n\exp\left[\sum_{i=1}^{n}s_iV(y_i)\right]. \qquad (7.29)$$

Solving equation (7.28) by iteration, we obtain

$$\tilde{U}_I^{L,R}(x_2,x_1) = \sum_{n=0}^{\infty}(\mathrm{i}k)^n\sum_{\{s_i=\pm1\}}[s_1,\ldots,s_n]_{x_1}^{x_2}\tilde{J}_{s_n}^{L,R}\cdots\tilde{J}_{s_1}^{L,R} \qquad (7.30)$$

where $\tilde{J}_{s_i}^{L,R}$ means $\tilde{J}_\pm^{L,R}$ for $s_i=\pm1$. (Note that $\tilde{J}_{s_i}^{L,R}$ with $s_i=+1$ is not $\tilde{J}_1^{L,R}$.)

We rewrite (7.24) in terms of $\tilde{U}_I^{L,R}$ as

$$\bar{G} = 2\langle\xi_-;L|\mathrm{e}^{-V(z)\tilde{J}_3^L}\tilde{U}_I^L(z,x)\mathrm{e}^{V(x)\tilde{J}_3^L}\tilde{Q}_-^L\mathrm{e}^{-V(x)\tilde{J}_3^L}$$
$$\times\tilde{U}_I^L(x,x')M\mathrm{e}^{V(x')\tilde{J}_3^R}\tilde{Q}_-^R\mathrm{e}^{-V(x')\tilde{J}_3^R}\tilde{U}_I^R(x',z')\mathrm{e}^{V(z')\tilde{J}_3^R}|\xi_-';R\rangle \qquad (7.31)$$

where we have also used (7.18). From (3.27) it follows that

$$e^{V(x)\tilde{J}_3^L}\tilde{Q}_-^L e^{-V(x)\tilde{J}_3^L} = e^{-V(x)/2}\tilde{Q}_-^L \tag{7.32a}$$

$$e^{V(x')\tilde{J}_3^R}\tilde{Q}_-^R e^{-V(x')\tilde{J}_3^R} = e^{-V(x')/2}\tilde{Q}_-^R. \tag{7.32b}$$

For arbitrary complex numbers α and α', the coherent states satisfy

$$e^{\alpha'\tilde{b}^{\dagger R}\tilde{b}^R}|\eta'; R\rangle = |e^{\alpha'}\eta'; R\rangle \qquad \langle\eta; L|e^{\alpha\tilde{b}^{\dagger L}\tilde{b}^L} = \langle e^{\alpha}\eta; L|. \tag{7.33}$$

Hence we have

$$e^{V(z')\tilde{J}_3^R}|\xi'_-; R\rangle = |\hat{\xi}'_-; R\rangle \qquad \langle\xi_-; L|e^{-V(z)\tilde{J}_3^L} = \langle\hat{\xi}_-; L| \tag{7.34}$$

where we have used definition (4.19). Inserting (7.32) and (7.34) into (7.31) gives

$$\bar{G} = 2e^{-[V(x)+V(x')]/2}\langle\hat{\xi}_-; L|\tilde{U}_I^L(z,x)\tilde{Q}_-^L\tilde{U}_I^L(x,x')M\tilde{Q}_-^R\tilde{U}_I^R(x',z')|\hat{\xi}'_-; R\rangle. \tag{7.35}$$

We substitute into (7.35) the expressions (7.17) for $\tilde{Q}_-^{L,R}$. Since the operators $\tilde{U}_I^{L,R}$ do not create or annihilate fermions, from (7.21) it follows that $\langle\hat{\xi}_-; L|\tilde{U}_I^L\tilde{c}^{\dagger L} = 0$ and $\tilde{c}^R\tilde{U}_I^R|\hat{\xi}'_-; R\rangle = 0$. Therefore, we obtain

$$\bar{G} = 2e^{-[V(x)+V(x')]/2}\langle\hat{\xi}_-; L|\tilde{U}_I^L(z,x)\tilde{c}^L\tilde{U}_I^L(x,x')M\tilde{c}^{\dagger R}\tilde{U}_I^R(x',z')|\hat{\xi}'_-; R\rangle. \tag{7.36}$$

Substituting (7.30) into (7.36) yields the expansion of the propagator in powers of k as

$$\begin{aligned}\bar{G} = 2e^{-[V(x)+V(x')]/2} \\ \times \sum_{n_1,n_2,n_3=0}^{\infty}\sum_{\{s_i,s'_i,s''_i=\pm1\}}(\mathrm{i}k)^{n_1+n_2+n_3}C(s_1,\ldots,s_{n_1}; s'_1,\ldots,s'_{n_2}; s''_1,\ldots,s''_{n_3}) \\ \times[s_1,\ldots,s_{n_1}]_{z'}^{x'}[s'_1,\ldots,s'_{n_2}]_{x'}^{x}[s''_1,\ldots,s''_{n_3}]_x^z\end{aligned} \tag{7.37}$$

where

$$\begin{aligned}C(s_1,\ldots,s_{n_1}; s'_1,\ldots,s'_{n_2}; s''_1,\ldots,s''_{n_3}) \\ \equiv \langle\hat{\xi}_-; L|\tilde{J}^L_{s''_{n_3}}\cdots\tilde{J}^L_{s''_1}\tilde{c}^L\tilde{J}^L_{s'_{n_2}}\cdots\tilde{J}^L_{s'_1}M\tilde{c}^{\dagger R}\tilde{J}^R_{s_{n_1}}\cdots\tilde{J}^R_{s_1}|\hat{\xi}'_-; R\rangle.\end{aligned} \tag{7.38a}$$

By using (7.18), we can also write

$$\begin{aligned}C(s_1,\ldots,s_{n_1}; s'_1,\ldots,s'_{n_2}; s''_1,\ldots,s''_{n_3}) \\ \equiv \langle\hat{\xi}_-; L|\tilde{J}^L_{s''_{n_3}}\cdots\tilde{J}^L_{s''_1}\tilde{c}^L M\tilde{J}^R_{s'_{n_2}}\cdots\tilde{J}^R_{s'_1}\tilde{c}^{\dagger R}\tilde{J}^R_{s_{n_1}}\cdots\tilde{J}^R_{s_1}|\hat{\xi}'_-; R\rangle.\end{aligned} \tag{7.38b}$$

We shall see in the next section how to calculate the coefficients C defined by (7.38).

8. Calculation of the expansion coefficients

Since the operator $\tilde{J}_\pm^{L,R}$ does not change the number of fermions, we can divide it into sectors with a fixed fermion number. Namely, we can write

$$\tilde{J}_\pm^L = \tilde{J}_\pm^{(0)L}\tilde{c}^L\tilde{c}^{\dagger L} + \tilde{J}_\pm^{(1)L}\tilde{c}^{\dagger L}\tilde{c}^L \tag{8.1a}$$

$$\tilde{J}_\pm^R = \tilde{J}_\pm^{(0)R}\tilde{c}^R\tilde{c}^{\dagger R} + \tilde{J}_\pm^{(1)R}\tilde{c}^{\dagger R}\tilde{c}^R \tag{8.1b}$$

with

$$\tilde{J}_+^{(j)L} \equiv -\tilde{b}^{\dagger L}\tilde{b}^{\dagger L}\tilde{b}^L - j\tilde{b}^{\dagger L} \qquad \tilde{J}_-^{(j)L} \equiv \tilde{b}^L \tag{8.2a}$$

$$\tilde{J}_+^{(j)R} \equiv -\tilde{b}^{\dagger R}\tilde{b}^R\tilde{b}^R - j\tilde{b}^R \qquad \tilde{J}_-^{(j)R} \equiv \tilde{b}^{\dagger R} \tag{8.2b}$$

where $j = 0$ or 1. The operators (8.2) are obtained from (7.17) by replacing $\tilde{c}^{\dagger L}\tilde{c}^{L}$ or $\tilde{c}^{\dagger R}\tilde{c}^{R}$ with the number j. In other words, $\tilde{J}_{\pm}^{(j)L,R}$ is the restriction of $\tilde{J}_{\pm}^{L,R}$ to eigenstates of $\tilde{c}^{\dagger L,R}\tilde{c}^{L,R}$ with eigenvalue j. Substituting (8.1) into (7.38*a*), and using (7.21), we obtain the expression

$$C(s_1,\ldots,s_{n_1};s'_1,\ldots,s'_{n_2};s''_1,\ldots,s''_{n_3}) = \langle\hat{\xi};L|\tilde{J}^{(0)L}_{s''_{n_3}}\cdots\tilde{J}^{(0)L}_{s''_1}\,\tilde{J}^{(1)L}_{s'_{n_2}}\cdots\tilde{J}^{(1)L}_{s'_1}\tilde{c}^{L}M\tilde{c}^{\dagger R}\tilde{J}^{(0)R}_{s_{n_1}}\cdots\tilde{J}^{(0)R}_{s_1}|\hat{\xi}';R\rangle \tag{8.3}$$

which is more convenient for practical calculations.

We must first calculate the coefficient (8.3) for $n_1 = n_2 = n_3 = 0$: namely,

$$C(;;) = \langle\hat{\xi}_-;L|\tilde{c}^{L}M\tilde{c}^{\dagger R}|\hat{\xi}'_-;R\rangle. \tag{8.4}$$

By using (7.22), we can rewrite this as

$$C(;;) = \langle\hat{h}|\tilde{c}^{L}M\tilde{c}^{\dagger R}|\hat{h}'\rangle \tag{8.5}$$

where $\hat{h} \equiv (1+\hat{\xi}_-)/(1-\hat{\xi}_-)$, $\hat{h}' \equiv (1+\hat{\xi}'_-)/(1-\hat{\xi}'_-)$. From (7.14) and (7.5) we have

$$\tilde{c}^{L} = \frac{1}{\sqrt{2}}(1+b^{\dagger})c \qquad \tilde{c}^{\dagger R} = \frac{1}{\sqrt{2}}(1+b)c^{\dagger}. \tag{8.6}$$

Substituting (8.6) into (8.5), and using (6.20), (6.19) and (6.4), we obtain

$$\begin{aligned} C(;;) &= \tfrac{1}{2}(1+\hat{h})(1+\hat{h}')\langle 0|\mathrm{e}^{\hat{h}b}cMc^{\dagger}\mathrm{e}^{\hat{h}'b^{\dagger}}|0\rangle \\ &= \frac{1}{2}(1+\hat{h})(1+\hat{h}')\sum_{n=0}^{\infty}\frac{(\hat{h}\hat{h}')^n}{(n!)^2}\langle 0|b^n cMc^{\dagger}(b^{\dagger})^n|0\rangle \\ &= \frac{1}{2}(1+\hat{h})(1+\hat{h}')\sum_{n=0}^{\infty}(\hat{h}\hat{h}')^n = \frac{1}{2}\frac{(1+\hat{h})(1+\hat{h}')}{1-\hat{h}\hat{h}'} = \frac{-1}{\hat{\xi}_-+\hat{\xi}'_-}. \end{aligned} \tag{8.7}$$

Multiplying this by the factor $2\mathrm{e}^{-[V(x)+V(x')]/2}$ (see equation (7.37)), we reproduce the previously obtained result (4.20) for the lowest order term. For simplicity, we shall denote the quantity (8.7) by C_0:

$$C_0 \equiv \frac{-1}{\hat{\xi}_-+\hat{\xi}'_-}. \tag{8.8}$$

It is important to note that C_0 has the property

$$\left(\frac{\partial}{\partial\hat{\xi}_-}\right)^{n}\left(\frac{\partial}{\partial\hat{\xi}'_-}\right)^{n'}C_0 = (n+n')!\,C_0^{n+n'+1}. \tag{8.9}$$

To calculate the coefficients (8.3) for general n_1, n_2, and n_3, we can make use of the relations (7.20). These relations yield, for an arbitrary operator A,

$$\langle\hat{\xi}_-;L|A\tilde{b}^{R}|\hat{\xi}'_-;R\rangle = \hat{\xi}'_-\langle\hat{\xi}_-;L|A|\hat{\xi}'_-;R\rangle \tag{8.10a}$$

$$\langle\hat{\xi}_-;L|\tilde{b}^{\dagger L}A|\hat{\xi}'_-;R\rangle = \hat{\xi}_-\langle\hat{\xi}_-;L|A|\hat{\xi}'_-;R\rangle \tag{8.10b}$$

$$\langle\hat{\xi}_-;L|A\tilde{b}^{\dagger R}|\hat{\xi}'_-;R\rangle = \frac{\partial}{\partial\hat{\xi}'_-}\langle\hat{\xi}_-;L|A|\hat{\xi}'_-;R\rangle \tag{8.10c}$$

$$\langle\hat{\xi}_-;L|\tilde{b}^{L}A|\hat{\xi}'_-;R\rangle = \frac{\partial}{\partial\hat{\xi}_-}\langle\hat{\xi}_-;L|A|\hat{\xi}'_-;R\rangle. \tag{8.10d}$$

If we define the differential operators

$$\hat{\mathcal{J}}_{+}^{(j)L} \equiv -\hat{\xi}_-^2\frac{\partial}{\partial\hat{\xi}_-} - j\hat{\xi}_- \qquad \hat{\mathcal{J}}_{-}^{(j)L} \equiv \frac{\partial}{\partial\hat{\xi}_-} \tag{8.11a}$$

$$\hat{\mathcal{J}}_{+}^{(j)R} \equiv -(\hat{\xi}'_-)^2\frac{\partial}{\partial\hat{\xi}'_-} - j\hat{\xi}'_- \qquad \hat{\mathcal{J}}_{-}^{(j)R} \equiv \frac{\partial}{\partial\hat{\xi}'_-} \tag{8.11b}$$

then from (8.3) and (8.10) we obtain†

$$C(s_1, \ldots, s_{n_1}; s'_1, \ldots, s'_{n_2}; s''_1, \ldots, s''_{n_3}) = \hat{\mathcal{J}}^{(0)L}_{s''_{n_3}} \ldots \hat{\mathcal{J}}^{(0)L}_{s''_1} \hat{\mathcal{J}}^{(1)L}_{s'_{n_2}} \ldots \hat{\mathcal{J}}^{(1)L}_{s'_1} \hat{\mathcal{J}}^{(0)R}_{s_1} \ldots \hat{\mathcal{J}}^{(0)R}_{s_{n_1}} C_0. \tag{8.12}$$

It is easy to calculate the terms of order k in (7.37) by using (8.12). From (8.12) and (8.9) we have

$$C(+1;;) = \hat{\mathcal{J}}^{(0)R}_{+} C_0 = -(\hat{\xi}'_-)^2 C_0^2 \tag{8.13a}$$

$$C(;+1;) = \hat{\mathcal{J}}^{(1)L}_{+} C_0 = \hat{\xi}_- \hat{\xi}'_- C_0^2 \tag{8.13b}$$

$$C(;;+1) = \hat{\mathcal{J}}^{(0)L}_{+} C_0 = -\hat{\xi}_-^2 C_0^2 \tag{8.13c}$$

and, similarly,

$$C(-1;;) = C(;-1;) = C(;;-1) = C_0^2. \tag{8.13d}$$

Substituting (8.13) into (7.37), we obtain the coefficient g_1 of equation (4.17) as

$$g_1 = 2\mathrm{i}\mathrm{e}^{-[V(x)+V(x')]/2} C_0^2 \{[-1]^z_{z'} - (\hat{\xi}'_-)^2[+1]^{x'}_{z'} + \hat{\xi}_- \hat{\xi}'_- [+1]^x_{x'} - \hat{\xi}_-^2 [+1]^z_x\} \tag{8.14}$$

which agrees with (4.24).

We can use the expression (8.3) directly for the calculation of higher-order coefficients. From (8.10) and (8.9) we obtain the useful formula

$$\langle \hat{\xi}_-; L|(\tilde{b}^{\dagger L})^m (\tilde{b}^L)^n \tilde{c}^L M \tilde{c}^{\dagger R} (\tilde{b}^{\dagger R})^{n'} (\tilde{b}^R)^{m'} |\hat{\xi}'_-; R\rangle = (n+n')! \hat{\xi}_-^m (\hat{\xi}'_-)^{m'} C_0^{n+n'+1}. \tag{8.15}$$

Therefore, the coefficients (8.3) are easily obtained if we express the product of operators $\tilde{J}$ in terms of $\tilde{b}$ and $\tilde{b}^\dagger$ in the normal-ordered form. This can be done with the help of the recursion relations

$$\tilde{J}^{(j)L}_{+} (\tilde{b}^{\dagger L})^m (\tilde{b}^L)^n = -(m+j)(\tilde{b}^{\dagger L})^{m+1} (\tilde{b}^L)^n - (\tilde{b}^{\dagger L})^{m+2} (\tilde{b}^L)^{n+1} \tag{8.16a}$$

$$\tilde{J}^{(j)L}_{-} (\tilde{b}^{\dagger L})^m (\tilde{b}^L)^n = m(\tilde{b}^{\dagger L})^{m-1} (\tilde{b}^L)^n + (\tilde{b}^{\dagger L})^m (\tilde{b}^L)^{n+1} \tag{8.16b}$$

$$(\tilde{b}^{\dagger R})^n (\tilde{b}^R)^m \tilde{J}^{(j)R}_{+} = -(m+j)(\tilde{b}^{\dagger R})^n (\tilde{b}^R)^{m+1} - (\tilde{b}^{\dagger R})^{n+1} (\tilde{b}^R)^{m+2} \tag{8.16c}$$

$$(\tilde{b}^{\dagger R})^n (\tilde{b}^R)^m \tilde{J}^{(j)R}_{-} = m\, (\tilde{b}^{\dagger R})^n (\tilde{b}^R)^{m-1} + (\tilde{b}^{\dagger R})^{n+1} (\tilde{b}^R)^m. \tag{8.16d}$$

It would be instructive to see the structure of the above expressions in terms of diagrams. Let us first think about the cases $n_1 = 0$ in (8.3). As an example, we consider the second-order coefficient $C(;+1;+1)$. By using (8.16), we can express $\tilde{J}^{(0)L}_{+} \tilde{J}^{(1)L}_{+}$ in the normal-ordered form as

$$\tilde{J}^{(0)L}_{+} \tilde{J}^{(1)L}_{+} = (\tilde{b}^{\dagger L})^2 + 3(\tilde{b}^{\dagger L})^3 \tilde{b}^L + (\tilde{b}^{\dagger L})^4 (\tilde{b}^L)^2. \tag{8.17}$$

Using the diagrammatic interpretation of $\tilde{J}^L_\pm$ shown in figure 15, and recalling the definition of $\tilde{J}^{(j)L}_\pm$, we can interpret relation (8.17) as in figure 16. Namely, each term in the normal-ordered form corresponds to a tree diagram. The root of each tree corresponds to the operator $\tilde{b}^L$, and each treetop corresponds to $\tilde{b}^{\dagger L}$. Substituting (8.17) into (8.3), and using (8.15), we obtain the coefficient $C(;+1;+1)$ as

$$\langle \hat{\xi}_-; L|\tilde{J}^{(0)L}_{+} \tilde{J}^{(1)L}_{+} \tilde{c}^L M \tilde{c}^{\dagger R} |\hat{\xi}'_-; R\rangle = C_0 \hat{\xi}_-^2 + 3C_0^2 \hat{\xi}_-^3 + 2C_0^3 \hat{\xi}_-^4. \tag{8.18}$$

The right-hand side of (8.18) can be interpreted as in figure 17. We can see that figure 17 almost faithfully preserves the structure of figure 16. The $\tilde{b}^{\dagger L}$ and $\tilde{b}^L$ in figure 16 are replaced by $\hat{\xi}_-$

† This is equivalent to the expansion formula derived in [16]. The correspondence between equation (5.26) of [16] and equation (8.12) of this paper is given by $\hat{J}^{(0)}_{\mp} = \hat{\mathcal{J}}^{(0)L}_{\pm}$, $\hat{J}'^{(0)}_{\mp} = \hat{\mathcal{J}}^{(0)R}_{\pm}$, and $\mathrm{e}^{W/2} \hat{J}^{(1)}_{\mp} \mathrm{e}^{-W/2} = \hat{\mathcal{J}}^{(1)L}_{\pm}$, together with $\mathrm{e}^{-W} = -\hat{\xi}_-$ and $\mathrm{e}^{-W'} = -\hat{\xi}'_-$.

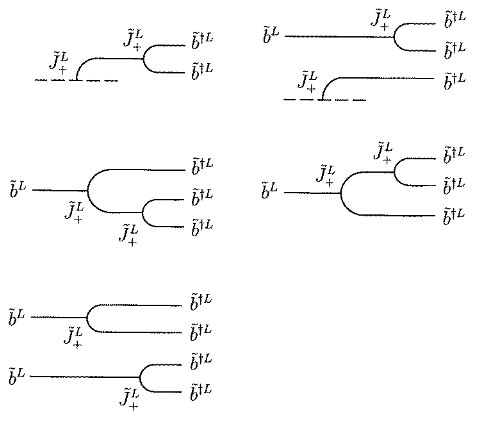

Figure 16. Schematic interpretation of equation (8.17). The broken line stands for the fermion line shown here for clarity, although fermion operators do not appear in the expression (8.17). Note that $\tilde{J}_+^{(1)L}$ includes the vertex involving the fermion, while $\tilde{J}_+^{(0)L}$ does not.

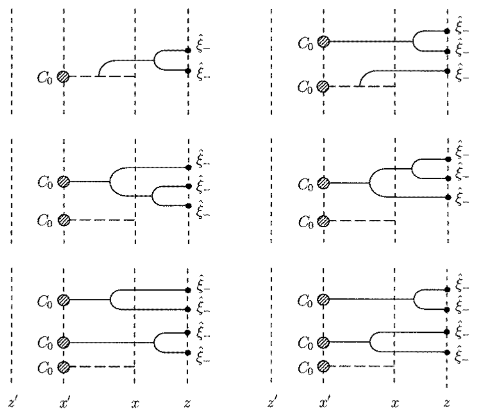

Figure 17. Graphical interpretation of equation (8.18).

and C_0, respectively. An additional C_0 is assigned to an endpoint of the fermion line. The important difference is that the two diagrams in the lowest row of figure 17 are distinguished, unlike the case in figure 16. Since we can rewrite (8.8) as

$$C_0 = \hat{\xi}'_+[1 + \hat{\xi}_-\hat{\xi}'_+ + (\hat{\xi}_-\hat{\xi}'_+)^2 + (\hat{\xi}_-\hat{\xi}'_+)^3 + \cdots] \tag{8.19}$$

the C_0 in figure 17 can be interpreted as in figure 18. From figure 18 we can also see the meaning of the property $(\partial/\partial\hat{\xi}_-)^n C_0 = n!C_0^n$; removing a $\hat{\xi}_-$ from the graph of C_0 gives rise to an additional C_0.

So far, we have been working in the interaction picture. We can recover the interpretation of the propagator as the 'sum over the paths' by going back to the 'Heisenberg' picture. To re-interpret figures 17 and 18 in the Heisenberg picture, we replace the $\hat{\xi}_\pm$ by $\xi_\pm$, and assign to each line the value of the free propagator given by figure 14(a). Then we can clearly see that the diagrams in figure 17 (with figure 18) are equivalent to the paths shown in figure 19.

It is straightforward to extend the diagrammatic interpretation to the cases $n_1 \neq 0$. A typical diagram corresponding to the right-hand side of (8.15) has the form shown in figure 20. The C_0, except the one assigned to the fermion line, can be arranged in an arbitrary order, and this gives the factor $(n + n')!$ in (8.15). Recall that we are using the frame with angle $+\pi/2$ in the region (z', x'), and $-\pi/2$ in (x, z). Since we are using different frames in these two regions, the diagram in figure 20 cannot be represented by a simple path as in figure 19. In figure 20, the trees in the region (z', x') are associated with the 'R' particles, whereas in (x', z) the trees correspond to the 'L' particles. These two different views are connected through the C_0.

Note that we could have obtained a more symmetric picture by rewriting (7.36) as

$$\begin{aligned}\bar{G}(x, x'; z, z'; \xi, \xi') &= 2\mathrm{e}^{-[V(x)+V(x')]/2} \\ &\quad\times\langle\hat{\xi}_-; L|\tilde{U}_I^L(z, x)\tilde{c}^L\tilde{U}_I^L(x, x_0)M\tilde{U}_I^R(x_0, x')\tilde{c}^{\dagger R}\tilde{U}_I^R(x', z')|\hat{\xi}'_-; R\rangle\end{aligned} \tag{8.20}$$

where x_0 is an arbitrary point between x' and x. (This expression is obtained by using $\tilde{U}_I^{L,R}(x, x_0)\tilde{U}_I^{L,R}(x_0, x') = \tilde{U}_I^{L,R}(x, x')$ and $M\tilde{U}_I^R = \tilde{U}_I^L M$.) If we had used (8.20) instead of (7.36), the diagram corresponding to each term of expansion would have looked like figure 21.

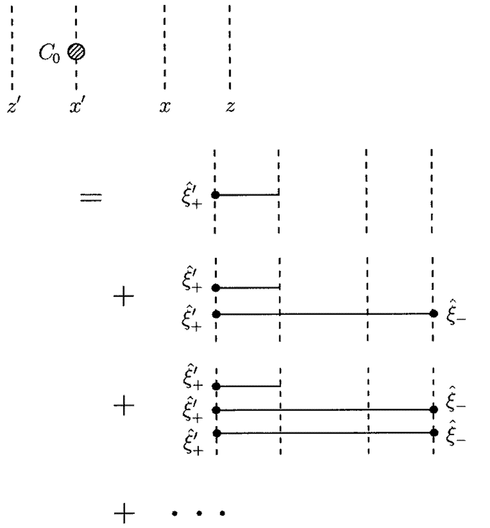

Figure 18. Graphical representation of C_0. (Note that the value of the propagator (figure 14) is not assigned to the lines in this figure.)

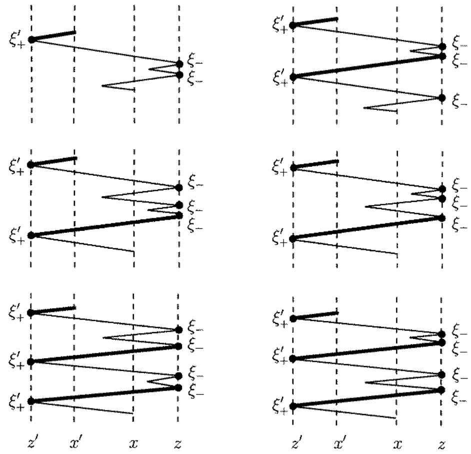

Figure 19. The tree diagrams in figure 17 are equivalent to the paths shown here. The heavy line is defined in the same way as figure 9 (with ξ_+ and ξ'_- replaced by ξ_- and ξ'_+).

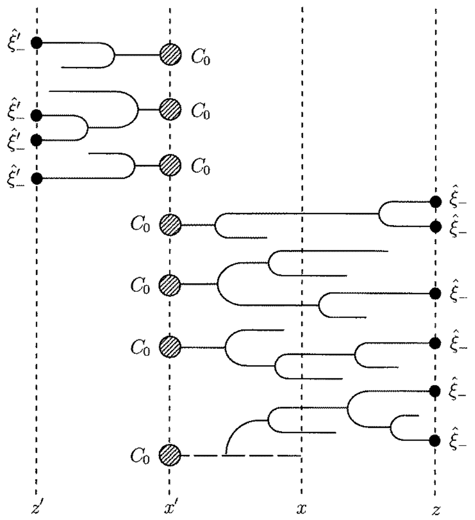

Figure 20. A typical tree diagram corresponding to the right-hand side of (8.15).

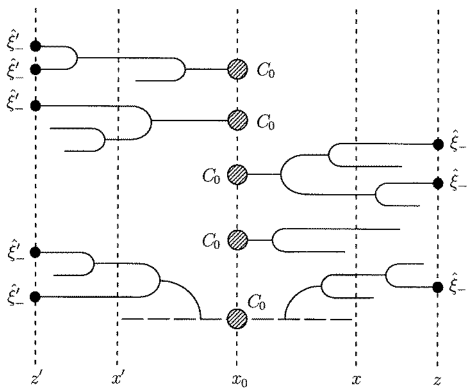

Figure 21. A diagram obtained from the expansion of (8.20).

9. Concluding remarks

Expression (7.37) is the low-energy expansion of the generalized propagator, where the coefficients C defined by (7.38) can be calculated by using (8.3), (8.16) and (8.15). From (1.9) we can see that the integrals $[s_1, s_2, \ldots, s_n]_{x_a}^{x_b}$, which are defined by (7.29), are expressed in terms of ψ_0 as

$$[s_1, s_2, \ldots, s_n]_{x_a}^{x_b} = \int \cdots \int_{x_a \leqslant y_1 \leqslant y_2 \leqslant \cdots \leqslant y_n \leqslant x_b} \mathrm{d}y_1 \ldots \mathrm{d}y_n \prod_{i=1}^{n} [\psi_0(y_i)]^{-2s_i}. \tag{9.1}$$

Thus the expansion coefficients of (7.37) are expressed in terms of the ground state wavefunction. The expansion of the original propagator G is obtained from (7.37) by taking the limit $z \to \infty$, $z' \to -\infty$. In this process, we need to assign appropriate values to ξ and ξ', corresponding to the behaviour of the potential $V(x)$ at $x \to +\infty$ and $x \to -\infty$.

Here let us remark on the types of the potential $V(x)$ admissible for this low-energy expansion formula. As mentioned in section 3, we can deal with three cases for the behaviour of $V(x)$ at $x \to \infty$: namely, $V(\infty) =$ finite, $V(\infty) = +\infty$, and $V(\infty) = -\infty$. In these cases, respectively, we set $\xi = 0$, $\xi = +1$, and $\xi = -1$ before taking the limit $z \to \infty$. For the expansion coefficients of (7.37) to remain finite in this limit, $V(x)$ should either converge sufficiently fast or diverge sufficiently fast at $x \to \infty$. It can be found from (7.37) and (8.12) that $V(x)$ should satisfy one of the following three conditions:

$$\left| \int_{x_{\min}}^{\infty} x^n [V(x) - V(\infty)] \mathrm{d}x \right| < \infty \qquad V(\infty) = \text{finite} \tag{9.2a}$$

$$\left| \int_{x_{\min}}^{\infty} x^n \mathrm{e}^{-V(x)} \mathrm{d}x \right| < \infty \qquad V(\infty) = +\infty \tag{9.2b}$$

$$\left| \int_{x_{\min}}^{\infty} x^n \mathrm{e}^{V(x)} \mathrm{d}x \right| < \infty \qquad V(\infty) = -\infty \tag{9.2c}$$

for any positive integer n and finite $x_{\min}$. (See [16] for a more detailed discussion.)

Note that these conditions are expressed in terms of the Fokker–Planck potential $V(x)$, and not the Schrödinger potential $V_S(x)$. From (1.4) it is obvious that (9.2*a*) is satisfied only if $V_S(\infty) = 0$. We can re-express (9.2*b*) in terms of ψ_0 as

$$\left| \int_{x_{\min}}^{\infty} x^n [\psi_0(x)]^2 \mathrm{d}x \right| < \infty. \tag{9.3}$$

This condition is satisfied if $V_S(\infty) = +\infty$ and $\psi_0(\infty) = 0$. Obviously (9.3) is also satisfied if, for example, $V_S(\infty)$ is finite and $\psi_0(x)$ decays exponentially at infinity. Thus, (9.2*b*) includes the cases of finite $V_S(\infty)$ as well as the cases $V_S(\infty) = +\infty$. The case (9.2*c*) has significance in quantum mechanics when the ground state is not a bound state. If there is no zero-energy bound state, the function $\psi_0(x)$ goes to $+\infty$ at $x \to +\infty$ or $x \to -\infty$. If $\psi_0(\infty) = +\infty$ we have $V(\infty) = -\infty$, whereas $V_S(\infty)$ is either finite or $+\infty$.

Similarly, there are three admissible cases for the behaviour of $V(x)$ at $x \to -\infty$, with the conditions analogous to (9.2). For $V(-\infty) =$ finite, $V(-\infty) = +\infty$, and $V(-\infty) = -\infty$, respectively, we set $\xi' = 0$, $\xi' = +1$, and $\xi' = -1$ before taking the limit $z' \to -\infty$ in (7.37). Considering both $x \to +\infty$ and $x \to -\infty$, we have $3 \times 3 = 9$ cases. In all these cases except two, we can let $z \to \infty$ and $z' \to -\infty$ in (7.37) with appropriate values of ξ and ξ'. The exceptional two cases are $V(\infty) = +\infty$ with $V(-\infty) = +\infty$ and $V(\infty) = -\infty$ with $V(-\infty) = -\infty$. In these two cases, we cannot simply take the limit of (7.37) to obtain the expansion of G. We can also deal with these cases, although a little trick is needed. This issue will be discussed in another paper. Here let us only mention that, in the case $V(\infty) = +\infty$ with $V(-\infty) = +\infty$, we can obtain the low-energy expansion of G by first expanding $1/\bar{G}$ in powers of k, and then taking the limit.

Finally, let us remark that this formalism is also applicable to the three-dimensional scattering by a spherically symmetric potential, which reduces to the radial problem in one dimension. The radial Schrödinger equation, which has the same form as (1.2), is restricted

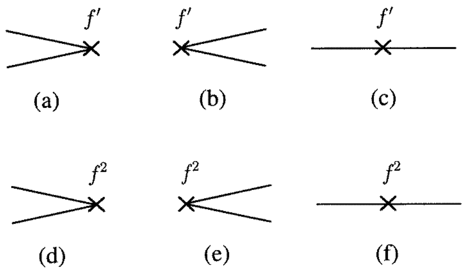

Figure A.1. Vertices obtained from figure 2 and equation (1.4).

within the half-line $0 \leqslant x < \infty$. Since we have $\psi_0(0) = 0$, the corresponding Fokker–Planck potential $V(x)$ goes to $+\infty$ at $x \to 0$. So we can obtain G by letting $z' \to 0$ with $\xi' = 1$.

Appendix A. Diagrammatic rules in terms of *f*

Inserting expression (1.4) into figures 2(*b*)–(*d*) gives the six scattering vertices shown in figure A.1. Let us calculate the first-order contribution to G_S that comes from (*a*)–(*c*). Corresponding to (*a*), (*c*), and (*b*) respectively, we have

$$\int_x^\infty \mathrm{d}y_1 G_0(x, y_1) f'(y_1) G_0(y_1, x') = \left(\frac{\mathrm{i}}{2k}\right)^2 \int_x^\infty \mathrm{d}y_1 \mathrm{e}^{\mathrm{i}k(y_1 - x)} \frac{\mathrm{d}f(y_1)}{\mathrm{d}y_1} \mathrm{e}^{\mathrm{i}k(y_1 - x')}$$

$$= -2\mathrm{i}k \left(\frac{\mathrm{i}}{2k}\right)^2 \int_x^\infty \mathrm{d}y_1 \mathrm{e}^{\mathrm{i}k(y_1 - x)} f(y_1) \mathrm{e}^{\mathrm{i}k(y_1 - x')} - \left(\frac{\mathrm{i}}{2k}\right)^2 f(x) \mathrm{e}^{\mathrm{i}k(x - x')} \qquad \text{(A.1}a\text{)}$$

$$\int_{x'}^x \mathrm{d}y_1 G_0(x, y_1) f'(y_1) G_0(y_1, x') = \left(\frac{\mathrm{i}}{2k}\right)^2 \int_{x'}^x \mathrm{d}y_1 \mathrm{e}^{\mathrm{i}k(x - y_1)} \frac{\mathrm{d}f(y_1)}{\mathrm{d}y_1} \mathrm{e}^{\mathrm{i}k(y_1 - x')}$$

$$= \left(\frac{\mathrm{i}}{2k}\right)^2 [f(x) - f(x')] \mathrm{e}^{\mathrm{i}k(x - x')} \qquad \text{(A.1}b\text{)}$$

$$\int_{-\infty}^{x'} \mathrm{d}y_1 G_0(x, y_1) f'(y_1) G_0(y_1, x') = \left(\frac{\mathrm{i}}{2k}\right)^2 \int_{-\infty}^{x'} \mathrm{d}y_1 \mathrm{e}^{\mathrm{i}k(x - y_1)} \frac{\mathrm{d}f(y_1)}{\mathrm{d}y_1} \mathrm{e}^{\mathrm{i}k(x' - y_1)}$$

$$= 2\mathrm{i}k \left(\frac{\mathrm{i}}{2k}\right)^2 \int_{-\infty}^{x'} \mathrm{d}y_1 \mathrm{e}^{\mathrm{i}k(x - y_1)} f(y_1) \mathrm{e}^{\mathrm{i}k(x' - y_1)} + \left(\frac{\mathrm{i}}{2k}\right)^2 f(x') \mathrm{e}^{\mathrm{i}k(x - x')}. \qquad \text{(A.1}c\text{)}$$

(Here we are assuming $x > x'$.) Adding (A.1*a*)–(A.1*c*), we have

$$\int_{-\infty}^\infty \mathrm{d}y_1 G_0(x, y_1) f'(y_1) G_0(y_1, x') = -2\mathrm{i}k \int_x^\infty \mathrm{d}y_1 G_0(x, y_1) f(y_1) G_0(y_1, x')$$

$$+2\mathrm{i}k \int_{-\infty}^{x'} \mathrm{d}y_1 G_0(x, y_1) f(y_1) G_0(y_1, x'). \qquad \text{(A.2)}$$

Thus we obtain the factors $-2\mathrm{i}kf$ and $+2\mathrm{i}kf$ for the left and right reflections, respectively.

The vertices (*d*)–(*f*) of figure A.1 are cancelled by a part of the second-order term

$$\int\int \mathrm{d}y_1 \mathrm{d}y_2 G_0(x, y_2) f'(y_2) G_0(y_2, y_1) f'(y_1) G_0(y_1, x') \qquad \text{(A.3)}$$

that comes from the singularity of $G_0(y_2, y_1)$ at $y_2 = y_1$. Indeed, since

$$\frac{\partial^2}{\partial y_1 \partial y_2} G_0(y_2, y_1) = -\delta(y_2 - y_1) + \text{ regular part} \qquad \text{(A.4)}$$

the delta function in (A.4) produces from (A.3) the term

$$-\int\int \mathrm{d}y_1\mathrm{d}y_2 G_0(x, y_2) f(y_2)\delta(y_2 - y_1) f(y_1) G_0(y_1, x')$$
$$= -\int \mathrm{d}y_1 G_0(x, y_1) f^2(y_1) G_0(y_1, x') \tag{A.5}$$

which cancels the contribution from (*d*)–(*f*).

Appendix B. Representation independence of (3.11)

Expression (3.11) does not depend on the particular form of the operators representing relations (3.4*a*) and (3.4*b*). The proof for the case $\xi = \xi' = 0$ is given in [14]. Here we show how to extend the proof to nonzero values of ξ and ξ'.

We define the generalized evolution operator $\bar{U}$ as

$$\bar{U}(x_b, x_a; \xi_b, \xi_a) \equiv D(\xi_b) U(x_b, x_a) D(-\xi_a) \tag{B.1}$$

where the operator D is defined as

$$D(\xi) \equiv \mathrm{e}^{\xi J_+}(1-\xi^2)^{J_3}\mathrm{e}^{\xi J_-} = \mathrm{e}^{\xi J_-}(1-\xi^2)^{-J_3}\mathrm{e}^{\xi J_+}. \tag{B.2}$$

(The last expression of (B.2) can be derived by using formulae (C.1) and (C.2) given in appendix C.) The meaning of the operator $D(\xi)$ is explained in [15]. This $\bar{U}$, just like U, satisfies the product rule

$$\bar{U}(x_c, x_b; \xi_c, \xi_b)\bar{U}(x_b, x_a; \xi_b, \xi_a) = \bar{U}(x_c, x_a; \xi_c, \xi_a). \tag{B.3}$$

Since $J_-\Psi_0 = Q_-Q_-\Psi_0 = 0$, we can rewrite (3.11) in terms of $\bar{U}$ as

$$\bar{G}(x, x'; z, z'; \xi, \xi') = \frac{1}{2}\frac{(\Psi_0^b, \bar{U}(z, x; \xi, 0)(Q_+ + Q_-)\bar{U}(x, x'; 0, 0)(Q_+ + Q_-)\bar{U}(x', z'; 0, \xi')\Psi_0^a)}{(\Psi_0^b, \bar{U}(z, z'; \xi, \xi')J_3\Psi_0^a)} \tag{B.4}$$

where $\Psi_0^a = (1-\xi'^2)^{-J_3}\Psi_0$ and $\Psi_0^b = (1-\xi^2)^{-J_3}\Psi_0$. It is obvious that both Ψ_0^a and Ψ_0^b satisfy the lowest-state condition:

$$Q_-\Psi_0^a = 0 \qquad Q_-\Psi_0^b = 0. \tag{B.5}$$

In fact, (B.4) is a more general expression than (3.11); expression (B.4) is valid for any Ψ_0^a and Ψ_0^b satisfying (B.5), as long as $(\Psi_0^b, J_3\Psi_0^a) \neq 0$. (Here we are allowing for the possibility that the lowest state satisfying (3.10) is not unique, as in the case of a reducible representation. If Ψ_0 is unique up to a constant factor, it is not necessary to distinguish between Ψ_0^a and Ψ_0^b in (B.4).)

As shown in [14], the evolution operator U, which is defined as the solution of (3.12), can be expressed in terms of the scattering coefficients τ, R_r, and R_l as

$$U(x_b, x_a) = \mathrm{e}^{-R_r(x_b,x_a)J_+}[\tau(x_b, x_a)]^{2J_3}\mathrm{e}^{R_l(x_b,x_a)J_-}. \tag{B.6}$$

Substituting (B.6) and (B.2) into (B.1), and using the commutation relations (3.5*a*), we can express $\bar{U}$ as

$$\bar{U}(x_b, x_a; \xi_b, \xi_a) = \mathrm{e}^{-\bar{R}_r(x_b,x_a;\xi_b,\xi_a)J_+}[\bar{\tau}(x_b, x_a; \xi_b, \xi_a)]^{2J_3}\mathrm{e}^{\bar{R}_l(x_b,x_a;\xi_b,\xi_a)J_-} \tag{B.7}$$

where

$$\bar{\tau}(x_b, x_a; \xi_b, \xi_a) = \frac{(1-\xi_b^2)^{1/2}(1-\xi_a^2)^{1/2}\tau}{1 - R_r\xi_b - R_l\xi_a - (\tau^2 - R_l R_r)\xi_b\xi_a} \tag{B.8a}$$

$$\bar{R}_r(x_b, x_a; \xi_b, \xi_a) = -\xi_b + (1-\xi_b^2)\frac{R_r + (\tau^2 - R_l R_r)\xi_a}{1 - R_r\xi_b - R_l\xi_a - (\tau^2 - R_l R_r)\xi_b\xi_a} \tag{B.8b}$$

$$\bar{R}_l(x_b, x_a; \xi_b, \xi_a) = -\xi_a + (1-\xi_a^2)\frac{R_l + (\tau^2 - R_l R_r)\xi_b}{1 - R_r\xi_b - R_l\xi_a - (\tau^2 - R_l R_r)\xi_b\xi_a}. \tag{B.8c}$$

(See [15] for details. The meaning and significance of $\bar{\tau}$, $\bar{R}_r$, and $\bar{R}_l$ are discussed therein.)

Now the representation-independence of (B.4) can be proved in the same way as for the case $\xi = \xi' = 0$. Using only the relations (3.4) (and (3.5)) and the expression (B.7), we can show that

$$\begin{aligned} Q_+\bar{U} &= \bar{\tau}^{-1}\bar{U}(Q_+ - \bar{R}_l Q_-) \qquad & Q_-\bar{U} &= \bar{\tau}^{-1}\bar{U}[\bar{R}_r Q_+ + (\bar{\tau}^2 - \bar{R}_l\bar{R}_r)Q_-] \\ \bar{U}Q_- &= \bar{\tau}^{-1}(Q_- - \bar{R}_r Q_+)\bar{U} \qquad & \bar{U}Q_+ &= \bar{\tau}^{-1}[\bar{R}_l Q_- + (\bar{\tau}^2 - \bar{R}_l\bar{R}_r)Q_+]\bar{U}. \end{aligned} \tag{B.9}$$

Substituting (B.9) into (B.4), and using (B.5) and (3.8), we have

$$\begin{aligned} \bar{G}(x, x'; z, z'; \xi, \xi') &= \frac{1 + \bar{R}_l(z, x; \xi, 0)}{\bar{\tau}(z, x; \xi, 0)} \frac{1 + \bar{R}_r(x', z'; 0, \xi')}{\bar{\tau}(x', z'; 0, \xi')} \\ &\quad \times \frac{1}{2} \frac{(\Psi_0^b, Q_-\bar{U}(z, x; \xi, 0)\bar{U}(x, x'; 0, 0)\bar{U}(x', z'; 0, \xi')Q_+\Psi_0^a)}{(\Psi_0^b, \bar{U}(z, z'; \xi, \xi')J_3\Psi_0^a)}. \end{aligned} \tag{B.10}$$

Using (B.7), (3.4), (3.5), (3.8) and (B.5), we can also show

$$(\Psi_0^b, Q_-\bar{U}Q_+\Psi_0^a) = 2\bar{\tau}(\Psi_0^b, \bar{U}J_3\Psi_0^a). \tag{B.11}$$

(See equation (5.11) of [14] for the intermediate steps of calculation.) From (B.10), (B.3) and (B.11), we obtain

$$\bar{G}(x, x'; z, z'; \xi, \xi') = \frac{1 + \bar{R}_l(z, x; \xi, 0)}{\bar{\tau}(z, x; \xi, 0)} \frac{1 + \bar{R}_r(x', z'; 0, \xi')}{\bar{\tau}(x', z'; 0, \xi')} \bar{\tau}(z, z'; \xi, \xi'). \tag{B.12}$$

Thus, (B.4) reduces to (B.12) irrespective of the particular choice of representation.

Appendix C. Derivation of (4.7) and (7.10)

Here we derive equations (4.7) and (7.10) at the same time, since they have the same structure. We use expression (3.19) and the algebraic formulae†

$$A^{J_3} J_\pm = J_\pm A^{J_3 \pm 1}, \tag{C.1}$$

$$e^{AJ_-} e^{BJ_+} = (1 + AB)^{-J_3} e^{BJ_+} e^{AJ_-} (1 + AB)^{-J_3}. \tag{C.2}$$

We can calculate, with $c \equiv \cos(\theta/2)$, $s \equiv \sin(\theta/2)$, $t \equiv \tan(\theta/2)$,

$$\begin{aligned} P(-\theta)e^{-\xi' J_+} &= e^{-tJ_-} c^{2J_3} e^{tJ_+} e^{-\xi' J_+} \\ &= e^{-tJ_-} e^{c^2(t-\xi')J_+} c^{2J_3} \\ &= [1 - tc^2(t-\xi')]^{-J_3} e^{c^2(t-\xi')J_+} e^{-tJ_-} [1 - tc^2(t-\xi')]^{-J_3} c^{2J_3} \\ &= [c^2(1 + t\xi')]^{-J_3} e^{c^2(t-\xi')J_+} e^{-tJ_-} (1 + t\xi')^{-J_3} \\ &= e^{[(t-\xi')/(1+t\xi')]J_+} [c^2(1 + t\xi')]^{-J_3} e^{-tJ_-} (1 + t\xi')^{-J_3} \\ &= e^{-\xi'_{-\theta} J_+} [c^2(1 + t\xi')]^{-J_3} e^{-tJ_-} (1 + t\xi')^{-J_3}. \end{aligned} \tag{C.3}$$

† See appendix B of [14].

(The symbol c here should not be confused with the fermion operator.) In the last line of (C.3), we used definition (4.8). We apply both sides of (C.3) to $|0\rangle$, which satisfies

$$J_-|0\rangle = 0 \qquad J_3|0\rangle = \frac{\nu}{2}|0\rangle. \tag{C.4}$$

Here the value of ν is $\frac{1}{2}$ for (4.7) (the vacuum of the a-boson), and 0 for (7.10) (the vacuum of the b-boson and the c-fermion). Then we have

$$P(-\theta)\mathrm{e}^{-\xi' J_+}|0\rangle = [\cos(\theta/2) + \xi' \sin(\theta/2)]^{-\nu} \mathrm{e}^{-\xi'_{-\theta} J_+}|0\rangle. \tag{C.5}$$

Therefore we obtain

$$\begin{aligned} \mathrm{e}^{-\xi' J_+}|0\rangle &= [\cos(\theta/2) + \xi' \sin(\theta/2)]^{-\nu} P(\theta)\mathrm{e}^{-\xi'_{-\theta} J_+}|0\rangle \\ &= [\cos(\theta/2) + \xi' \sin(\theta/2)]^{-\nu} \mathrm{e}^{-\xi'_{-\theta} J_{+,\theta}} P(\theta)|0\rangle \\ &= [\cos(\theta/2) + \xi' \sin(\theta/2)]^{-\nu} \mathrm{e}^{-\xi'_{-\theta} J_{+,\theta}}|0; \theta\rangle. \end{aligned} \tag{C.6}$$

Setting $\nu = \frac{1}{2}$ and $\nu = 0$ in (C.6), we obtain, respectively, (4.7a) and the first equation of (7.10). The relations for the bras can be derived in the same way.

Appendix D. Derivation of (7.5)

We apply both sides of (C.3) (with P and J replaced by P^R and J^R) to the state $c^\dagger|0\rangle$, where $|0\rangle$ is the vacuum of the b-boson and the c-fermion. Using the relations

$$J_-^R c^\dagger|0\rangle = 0 \qquad J_3^R c^\dagger|0\rangle = \tfrac{1}{2} c^\dagger|0\rangle \tag{D.1}$$

we can derive

$$P^R(-\theta)\mathrm{e}^{\xi' b^\dagger} c^\dagger|0\rangle = \mathrm{e}^{\xi'_{-\theta} b^\dagger} \frac{1}{\cos(\theta/2) + \xi' \sin(\theta/2)} c^\dagger|0\rangle. \tag{D.2}$$

Applying $P^R(\theta)$ to both sides of (D.2), we obtain

$$\mathrm{e}^{\xi' b^\dagger} c^\dagger|0\rangle = \mathrm{e}^{\xi'_{-\theta} b^{\dagger R}_\theta} \frac{1}{\cos(\theta/2) + \xi' \sin(\theta/2)} c^{\dagger R}_\theta|0; \theta\rangle. \tag{D.3a}$$

In a similar way, we can derive

$$\langle 0|c\mathrm{e}^{\xi b} = \langle 0; \theta|c^L_\theta \frac{1}{\cos(\theta/2) - \xi \sin(\theta/2)} \mathrm{e}^{\xi_\theta b^L_\theta}. \tag{D.3b}$$

We extend the definition of the coherent states to include the fermion as

$$|\xi', \gamma'\rangle \equiv \mathrm{e}^{b^\dagger \xi'} \mathrm{e}^{c^\dagger \gamma'}|0\rangle = \mathrm{e}^{b^\dagger \xi'}(1 + c^\dagger \gamma')|0\rangle \tag{D.4a}$$

$$\langle \xi, \gamma| \equiv \langle 0|\mathrm{e}^{\gamma c}\mathrm{e}^{\xi b} = \langle 0|(1 + \gamma c)\mathrm{e}^{\xi b} \tag{D.4b}$$

where γ and γ' are Grassmann numbers, satisfying

$$\gamma^2 = (\gamma')^2 = 0 \qquad \{\gamma, c\} = \{\gamma, c^\dagger\} = \{\gamma', c\} = \{\gamma', c^\dagger\} = 0. \tag{D.5}$$

These coherent states satisfy

$$\begin{aligned} &b|\xi', \gamma'\rangle = \xi'|\xi', \gamma'\rangle \qquad c|\xi', \gamma'\rangle = \gamma'|\xi', \gamma'\rangle \\ &b^\dagger|\xi', \gamma'\rangle = \frac{\partial}{\partial \xi'}|\xi', \gamma'\rangle \qquad c^\dagger|\xi', \gamma'\rangle = -\frac{\partial}{\partial \gamma'}|\xi', \gamma'\rangle \\ &\langle \xi, \gamma|b^\dagger = \xi\langle \xi, \gamma| \qquad \langle \xi, \gamma|c^\dagger = \gamma\langle \xi, \gamma| \\ &\langle \xi, \gamma|b = \frac{\partial}{\partial \xi}\langle \xi, \gamma| \qquad \langle \xi, \gamma|c = \frac{\partial}{\partial \gamma}\langle \xi, \gamma|. \end{aligned} \tag{D.6}$$

From (D.4), (D.3) and (7.10) we find

$$|\xi', \gamma'\rangle = \mathrm{e}^{b_\theta^{\dagger R}\xi'_{-\theta}}\left(1 + \frac{c_\theta^{\dagger R}\gamma'}{\cos(\theta/2) + \xi' \sin(\theta/2)}\right)|0;\theta\rangle \tag{D.7a}$$

$$\langle \xi, \gamma| = \langle 0;\theta|\left(1 + \frac{\gamma c_\theta^L}{\cos(\theta/2) - \xi \sin(\theta/2)}\right)\mathrm{e}^{\xi_\theta b_\theta^L}. \tag{D.7b}$$

If we define

$$\gamma_\theta \equiv \frac{\gamma}{\cos(\theta/2) - \xi \sin(\theta/2)} \qquad \gamma'_{-\theta} \equiv \frac{\gamma'}{\cos(\theta/2) + \xi' \sin(\theta/2)} \tag{D.8}$$

then equations (D.7) read

$$|\xi', \gamma'\rangle = \mathrm{e}^{b_\theta^{\dagger R}\xi'_{-\theta}}\mathrm{e}^{c_\theta^{\dagger R}\gamma'_{-\theta}}|0;\theta\rangle \qquad \langle \xi, \gamma| = \langle 0;\theta|\mathrm{e}^{\gamma_\theta c_\theta^L}\mathrm{e}^{\xi_\theta b_\theta^L}. \tag{D.9}$$

This is an extension of (7.10). Extending the notation of (7.11), we may write (D.9) as

$$|\xi', \gamma'\rangle = |\xi'_{-\theta}, \gamma'_{-\theta};\theta\rangle \qquad \langle \xi, \gamma| = \langle \xi_\theta, \gamma_\theta;\theta|. \tag{D.10}$$

Hence it follows that

$$\begin{aligned}
&b_\theta^R|\xi', \gamma'\rangle = \xi'_{-\theta}|\xi', \gamma'\rangle \qquad c_\theta^R|\xi';\gamma'\rangle = \gamma'_{-\theta}|\xi', \gamma'\rangle \\
&b_\theta^{\dagger R}|\xi', \gamma'\rangle = \frac{\partial}{\partial \xi'_{-\theta}}|\xi', \gamma'\rangle \qquad c_\theta^{\dagger R}|\xi', \gamma'\rangle = -\frac{\partial}{\partial \gamma'_{-\theta}}|\xi', \gamma'\rangle \\
&\langle \xi, \gamma|b_\theta^{\dagger L} = \xi_\theta\langle \xi, \gamma| \qquad \langle \xi, \gamma|c_\theta^{\dagger L} = \gamma_\theta\langle \xi, \gamma| \\
&\langle \xi, \gamma|b_\theta^L = \frac{\partial}{\partial \xi_\theta}\langle \xi, \gamma| \qquad \langle \xi, \gamma|c_\theta^L = \frac{\partial}{\partial \gamma_\theta}\langle \xi, \gamma|.
\end{aligned} \tag{D.11}$$

From (4.8) and (D.8) we have

$$\begin{aligned}
\frac{\partial}{\partial \xi_\theta} &= \left(\cos\frac{\theta}{2} - \xi \sin\frac{\theta}{2}\right)^2\frac{\partial}{\partial \xi} - \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - \xi \sin\frac{\theta}{2}\right)\gamma\frac{\partial}{\partial \gamma} \\
\frac{\partial}{\partial \xi'_{-\theta}} &= \left(\cos\frac{\theta}{2} + \xi' \sin\frac{\theta}{2}\right)^2\frac{\partial}{\partial \xi'} + \sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} + \xi' \sin\frac{\theta}{2}\right)\gamma'\frac{\partial}{\partial \gamma'} \\
\frac{\partial}{\partial \gamma_\theta} &= \left(\cos\frac{\theta}{2} - \xi \sin\frac{\theta}{2}\right)\frac{\partial}{\partial \gamma} \qquad \frac{\partial}{\partial \gamma'_{-\theta}} = \left(\cos\frac{\theta}{2} + \xi' \sin\frac{\theta}{2}\right)\frac{\partial}{\partial \gamma'}.
\end{aligned} \tag{D.12}$$

We substitute (4.8), (D.8) and (D.12) into the right-hand sides of (D.11). Then, we use (D.6) to replace ξ', γ', etc by b, c, etc. The resulting relations are equations (7.5) applied to the coherent states. Since these relations are satisfied for arbitrary coherent states, we find that equations (7.5) must hold by themselves.

References

[1] Chadan K and Sabatier P C 1989 *Inverse Problems in Quantum Scattering Theory* 2nd edn (New York: Springer)
[2] Faddeev L D 1995 *Acta Appl. Math.* **39** 69
[3] Gesztesy K A and Simon B 1996 *Acta Math.* **176** 49
[4] Danielyan A A and Levitan B M 1991 *Math. USSR Izv.* **36** 487
[5] Klaus M 1988 *Inverse Problems* **4** 505
[6] Witten E 1981 *Nucl. Phys.* B **185** 513
[7] Grosse H 1991 *Recent Developments in Quantum Mechanics* ed A Boutet de Monvel *et al* (Dordrecht: Kluwer) p 299
[8] Nelson E 1966 *Phys. Rev.* **150** 1079
[9] Parisi G and Wu Y 1981 *Sci. Sin.* **24** 483
[10] Bernstein M and Brown L S 1984 *Phys. Rev. Lett.* **52** 1933
[11] Jauslin H R 1988 *J. Phys. A: Math. Gen.* **21** 2337

[12] Risken H 1984 *The Fokker–Planck Equation* (Berlin: Springer)
[13] van Kampen N G 1981 *Stochastic Processes in Physics and Chemistry* (Amsterdam: North-Holland)
[14] Miyazawa T 1995 *J. Math. Phys.* **36** 5643
[15] Miyazawa T 1998 *J. Math. Phys.* **39** 2035
[16] Miyazawa T 1999 *J. Math. Phys.* **40** 838
[17] Slansky R 1987 *Les Houches, Session 44 1985* ed P Ramond and R Stora (Amsterdam: North-Holland) p 905
[18] Freund P G O 1986 *Introduction to Supersymmetry* (Cambridge: Cambridge University Press)